TOPIC PROPOSAL: THE STRUCTURE OF FINITE GROUPS

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DISCUSSED WITH GEORGE GLAUBERMAN

The typical college course in group theory covers groups, homomorphisms, normal subgroups, quotients, and group actions. This is followed by two basic theorems about finite groups: Sylow's theorem and the Jordan-Holder theorem. Sylow's theorem is the starting point of the *arithmetic* structure of groups: it relates the factorization of a natural number n to subgroups in a group of order n. The Jordan-Holder theorem is the starting point of the *normal* structure of groups: the use of normal subgroups and quotient groups to study a group's structure. (For more discussion on the arithmetic and normal structure, refer [1]).

The Jordan-Holder theorem relates arbitrary finite groups to a restricted class of finite groups called finite simple groups. A **simple group** is a nontrivial group that has no proper nontrivial normal subgroup. Unlike what the name suggests, *simple* here does not mean straightforward, gullible, or easy. Rather, simple means *unbreakable* or *indecomposable*.

By the eighties, finite group theorists had completed (modulo a snag that was finally resolved in 2004) the classification of all finite simple groups. ([2] is an excellent expository article on the classification and its history). Unlike what some believe, this does not render trivial the study of questions about *all finite groups*, since there are many different ways that finite simple groups can be "put together." For instance, all finite p-groups are obtained by piecing together simple groups of order p. However, there are many nontrivial questions about the structure of finite p-groups. In fact, classifying all finite groups, or even all finite p-groups, may not be meaningful in any sense because there are just too many types.

With no large-scale classification in the works, group theorists are trying to add that extra bit to our knowledge of how finite groups behave. They're also building on the effort of the classification by helping out in other areas of mathematics. Some examples worth mentioning here:

- Geometric group theorists are interested in studying group actions on manifolds. Often, the study of such actions boils down to questions about p-groups and simple groups.
- Pro-p-groups, and profinite groups in general, are of tremendous importance to questions in number theory, algebraic geometry, and Galois theory. For instance, the inverse Galois problem can be phrased as a question of computing the finite quotients of a given profinite group. Solving many of these questions requires exploration of questions involving finite groups.
- p-local classifying spaces of finite groups are important objects in algebraic topology. The recent development of the notion of fusion system allows for the construction of classifying spaces of fusion systems, which is a generalization of p-local classifying spaces of finite groups. The construction of fusion systems is inspired by the fusion theorems proved by Alperin and others as work towards the classification. (More on this in section 3.1; also see [8] for an introduction to fusion systems).

I hope to pursue research in group theory. My topic proposal describes some old work (1900-1975) in group theory, done originally with the classification in mind, that might help with such research.

My topic covers material largely from Chapters 5 to 8 of Gorenstein's book *Finite Groups* ([3]).

1. Groups of prime power order

1.1. Characteristic subgroups of finite p-groups. Let p be a prime. A finite p-group is a group whose order is a power of p.[1] A famous elementary result is that every finite p-group is nilpotent: in particular, every nontrivial finite p-group has a nontrivial center.

There are many important characteristic subgroups of finite p-groups, many of them defined through subgroup-defining functions: they're defined in a unique way starting from the big group. Examples include the center, commutator subgroup, members of the lower central series, members of the upper central series,

[1] The trivial group is a finite p-group as well, but people often implicitly mean "nontrivial" when talking about finite p-groups.

the **Frattini subgroup** (defined as the intersection of all maximal subgroups), and members of the Frattini series (a series obtained by taking Frattini subgroups successively). Also important are the omega and agemo subgroups: the subgroup $\Omega_j(P)$ of a p-group P is defined as the subgroup generated by all $x \in P$ for which the order of x divides p^j. $\mho^j(P)$ is defined as the subgroup generated by all elements of the form x^{p^j}.

Why the plethora of subgroup-defining functions? The hope is to find smaller characteristic subgroups inside the big group whose structure gives information about the structure of the whole group.

1.2. **Measures of size.** How big is a p-group? The naive measure of size is the order. A group of order p^n thus has size p^n. However, size on its own conveys little information about complexity. For instance, the structure of the cyclic group of order p is similar for all primes p, and these groups might be thought of as equally complex in some sense, even though their sizes differ widely. This suggests that the measure of size is not p^n, but the exponent n.

However, for large values of n, the value of p matters too. When $n = 1$, all groups of order p^1 can be studied under the same umbrella: cyclic groups of prime order. The same holds for $n = 2, 3$ (with $p = 2$ behaving slightly differently) and to a somewhat lesser extent, for $n = 4$. But for $n \geq 5$, the number of groups of order p^n, and their nature and behavior, depends very much on p.

We have two measures of size so far: p^n and n. Other measures of size look at the length of various series associated with the group. For instance, we can consider the nilpotence class (the length of the lower or upper central series), the solvable length (the length of the derived series), or the Frattini length (the length of the Frattini series). These measures are related in strange ways, and none of them alone suffices. For instance:

- A nontrivial p-group is abelian if and only if it has nilpotence class one, if and only if it has solvable length one. Thus, groups of small nilpotence class and small solvable length can be thought of as being close to abelian.
- A group of nilpotence class c has solvable length *at most* c, but the solvable length provides no bound on the nilpotence class.
- A nontrivial p-groups is elementary abelian[2] if and only if it has Frattini length one.

Yet another measure of size is the exponent. The exponent of a finite group is the smallest d such that the order of every element divides d. The exponent of a p-group is of the form p^e for some natural number e, so we may choose either e or p^e as the size measure.

Here are some interesting and unexpected facts:

- When $p = 2$, any group of exponent p is abelian. When $p = 3$, any group of exponent p has nilpotence class at most three. When $p \geq 5$, there is no bound on the nilpotence class of a p-group of exponent p.
- A group of order p^n and nilpotence class $n - 1$ (also called a **maximal class group**) is generally thought of as being very far from abelian. However, we can find maximal class groups with abelian subgroups of index p. For instance, the dihedral group of order 2^n, or the wreath product of two cyclic p-groups, are maximal class groups, yet have abelian subgroups of prime index. In fact, any 2-group of maximal class has a cyclic maximal subgroup (this follows from the classification of all 2-groups of maximal class, see for instance [3, Theorem 5.4.5, Page 194–195]).

1.3. **Coprime automorphisms and Thompson's critical subgroup.** For simplicity of expression, we introduce a term. A subgroup H of a finite group G is termed a **coprime automorphism-faithful subgroup** if, for any non-identity automorphism σ of G of order relatively prime to the order of G, such that $\sigma(H) = H$, the restriction of σ to H is not the identity map. In other words, non-identity automorphisms of coprime order, when they restrict, restrict to non-identity maps.

Thompson was interested in whether a small coprime automorphism-faithful *characteristic* subgroup can be found. If H is both coprime automorphism-faithful and characteristic, we get a homomorphism, by restriction:

$$\mathrm{Aut}(G) \to \mathrm{Aut}(H)$$

[2]An **elementary abelian group** is a direct product of isomorphic cyclic groups of prime order; equivalently, it is the additive group of a vector space over a finite field.

where the kernel contains no non-identity element of order coprime to that of G. Thompson managed to prove, in his paper with Feit on the odd-order theorem ([4, Chapter 2, Lemma 8.2, Page 795]) that every finite p-group has such a coprime automorphism-faithful characteristic subgroup of nilpotence class two. Nilpotence class two groups are very close to abelian groups: they are "small." Thompson used the faithful action of the coprime automorphism group on this small group to get strong restrictions on the nature of the coprime automorphism group.

The subgroup constructed by Thompson has many other remarkable properties as well. These properties were abstracted by Gorenstein ([3, Theorem 5.3.11, Page 185–186]), and a subgroup satisfying these properties is termed a **critical subgroup**.

The typical temptation when trying to build a characteristic subgroup with certain properties is to look for a subgroup-defining function: something that pinpoints a subgroup uniquely. So, one might look at the omega series, the agemo series, the Frattini series, the lower and upper central series, or the derived series. Thompson's insight was to *not* yield to this temptation. Rather, his construction of a critical subgroup relies on certain *choices*, and in general, there may be more than one critical subgroup. As yet, there is no known way of *canonically* picking a critical subgroup, despite the fact that critical subgroups are all characteristic. In fact, Thompson's procedure does not even yield all the possible critical subgroups.

1.4. **Abelian to cyclic: groups in a real-life act.** Students of group cohomology may recognize the condition for finite groups: *every abelian subgroup is cyclic*. This is precisely the condition for having periodic cohomology, and is relevant to topological questions such as group actions on spheres. The question can be broken into two parts using Sylow's theorem: find the p-groups with this property, and then, find all groups which have p-Sylow subgroups within that list of p-groups with the property. It turns out that when p is odd, a p-group in which every abelian subgroup is cyclic must itself by cyclic, and for $p = 2$, the group is either cyclic or generalized quaternion.[3]

Thompson's critical subgroup theorem, and many related discoveries that appeared in his paper with Feit on the odd-order theorem ([4]), can be used to answer more general versions of this question. For instance, it is possible to considerably simplify the proof of the classification of all p-groups in which *every abelian characteristic subgroup is cyclic* (originally done by Philip Hall), and all p-groups in which *every abelian normal subgroup is cyclic*.

1.5. **Differentiate and exponentiate.** The last thing you might expect to be able to do with a finite group is take logarithms, exponentials, and tangent spaces. And yet, it turns out that for a restricted class of finite p-groups (the subgroup generated by any three elements must have class at most $p-1$), there exist "Lie rings" corresponding to the groups, with a bijective map from the Lie ring to the group called the "exponential". This exponential behaves much the same way as the usual exponential for a nilpotent Lie group (such as a group of upper-triangular matrices). This establishes a correspondence between a restricted class of p-groups and a restricted class of p-Lie rings, called the Lazard correspondence. Here are some interesting observations about the correspondence:

- The exponential map is a bijection that preserves orders of elements, puts Lie subrings in correspondence with subgroups, and puts Lie ideals in correspondence with normal subgroups. It preserves automorphism groups as well.
- For abelian groups, the corresponding Lie ring is abelian with the same additive group. In other words, the Lie bracket is zero.
- For odd primes, the Lie ring corresponding to a group of class two is defined with the Lie bracket being the commutator in the group-theoretic sense.
- When the group has exponent p, the Lie ring is a Lie algebra over the prime field. This allows us to do "algebraic geometry" over the prime field.

The Lazard correspondence is more than a curiosity. Just as taking logarithms helps in converting messy arithmetic computation involving multiplication to relatively easier problems of addition, the Lazard correspondence helps escape the messy noncommutativity of groups and pass to the more pleasant and

[3]A **generalized quaternion group** of order 2^{n+1} is a group with a presentation $\langle a^{2^n} = 1, x^2 = a^{2^{n-1}}, xax^{-1} = a^{-1}\rangle$, $n \geq 2$. All generalized quaternion groups occur as subgroups of the skew field of Hamiltonian quaternions. The famed example is the quaternion group $\{\pm 1, \pm 1, \pm j, \pm k\}$ of order eight, obtained by setting $n = 2$.

tractable Lie rings. The Lazard Lie ring of a group of class two was used by Bender (in [5]) to simplify a computationally involved proof of Thompson about signalizers.

2. Solvable groups

2.1. **Schur-Zassenhaus and the shadow of odd order.** A subgroup H of a group G is termed a **Hall subgroup** if the order and index of H are relatively prime. Schur and Zassenhaus proved that any normal Hall subgroup has a permutable complement: in other words, if H is normal Hall in G, G splits as a semidirect product involving H. The proof has many variations, some using transversals, some making a direct appeal to group cohomology. They further proved that if H is normal Hall in G, and either H or G/H is solvable, then any two complements to H in G are conjugate.[4]

This "either one or the other is solvable" assumption may seem weak. However, given two coprime numbers, at least one of them is odd, and the celebrated odd-order theorem of Feit and Thompson ([4]) shows that any group of odd order is solvable. Thus, Zassenhaus' conclusion actually holds for any normal Hall subgroup. Unfortunately, the proof of the Feit-Thompson theorem is fairly involved (255 pages) and it feels bad to invoke such a big theorem to prove a relatively minor result.

The Schur-Zassenhaus theorem has many applications, one of them being a "Sylow's theorem with operators": a version of Sylow's theorem that measures all the Sylow subgroups invariant under the action of a coprime group of automorphisms. The usual existence, conjugacy, and domination statements hold.

3. Fusion and transfer

3.1. **Alperin's fusion: hopping step by step locally.** Fusion, despite its grand-sounding name, is about a simple question: given two elements in a big group that are conjugate, are they conjugate in some intermediate subgroup? And more generally, can we hop from one element to another using conjugates within intermediate subgroups?

This question is related to the idea of "local analysis". Suppose G is a finite group with possibly many primes dividing it. Local analysis basically says that certain behavior inside the big group G can be studied by looking at a number of smaller subgroups dependent on a prime p, along with information that describes how they piece together. In the typical scenario, these smaller subgroups are **p-local subgroups**: the normalizers of p-subgroups of the whole group. Local analysis was originally developed to study the local structure of simple groups, or candidates for simple groups. Those who've played with Sylow numbers to prove that certain groups aren't simple may think of local analysis as a much more advanced tool that works partly in that direction.

Alperin's fusion theorem ([6]) proves a certain result about selecting a family of subgroups of a p-Sylow subgroup, such that two elements (or more generally, subsets) are conjugate in the whole group if and only if there is a way from one element to another by conjugation using elements in the normalizers of this family of subgroups. In other words, it says that two subsets are conjugate *globally in the group* if and only if they are conjugate *locally, via normalizers of p-subgroups.* (For a more detailed treatment of local analysis, refer [7, Chapter 1]).

People working on the classification of finite simple groups came across a few stubborn examples of collections of local information that didn't give a group, but for which it was very hard to prove that there was no group behind them. It turns out that these collections of local information were actually something close to groups: they have been christened *fusion systems.* For any finite simple group and any prime p, we get a p-fusion system, but there exist p-fusion systems that arise through other means.

Algebraic topologists have long been interested in p-local classifying spaces. The p-local classifying space of a group is the p-localization of its classifying space. People working on fusion systems came up with a notion of the classifying space corresponding to a p-fusion system, such that if we take the p-fusion system $\mathcal{F}$ corresponding to a finite group G, the classifying space corresponding to $\mathcal{F}$ is equal to the p-local classifying space corresponding to G. (For a quick introduction to fusion systems, refer to [8]).

[4]Who contributed what part to the proof is unclear. It is definitely true that Schur set the ball rolling by proving existence, at least in the abelian case, and Zassenhaus stated the full theorem in his textbook later.

3.2. **The Focal Subgroup Theorem.** The Focal Subgroup Theorem states something seemingly uninteresting: if P is a p-Sylow subgroup of G, then the intersection of P with $[G, G]$ equals the subgroup generated by xy^{-1}, where $x, y \in P$ and are conjugate in G. This theorem has many proofs, the most typical being a proof using the transfer homomorphism. (The theorem, along with many related facts, were proved by Donald Higman as part of his doctoral thesis, [9]).

The Focal Subgroup Theorem has many consequences. For instance, define a subgroup H of a group G to be a **conjugacy-closed subgroup** of G if any two elements of H that are conjugate in G are conjugate in H. In other words, fusion in G is contained completely inside H.

The Focal Subgroup Theorem, combined with results about conjugacy and normalizers (such as Alperin's fusion theorem), yields that if an abelian Sylow subgroup S of G is in the center of its normalizer, it has a **normal complement**: a normal subgroup N of G such that $NS = G$ and $N \cap S$ is trivial. This result is called Burnside's normal p-complement theorem. More work in this direction shows that if S is any conjugacy-closed Sylow subgroup of G, S has a normal complement in G. A slightly more elaborate version of this result is called the Frobenius normal p-complement theorem.

Here are some examples of applications of Burnside's and Frobenius' normal p-complement theorems:

- Suppose p is the least prime divisor of the order of a finite group G. if G has a cyclic p-Sylow subgroup P, then P has a normal complement in G.
- Suppose G is a group in which every Sylow subgroup is cyclic. Then, G is solvable. The original proof used order computations involving elements, but the proof using the normal p-complement theorems is more intuitive.
- If G is a simple non-abelian group, either 12 divides the order of G or the cube of the smallest prime divisor of the order of G divides the order of G.

4. More powerful machinery

4.1. **Replacement theorems, and more normal complement theorems.** The next level of normal complement theorems, proved by Thompson and Glauberman, required a new array of techniques. We encountered Thompson's out-of-the-box thinking in the critical subgroup construction earlier.[5]

Here, we see *replacement*, yet another idea of Thompson. The idea is: *if at first you pick a bad subgroup, replace it with a better one.* In [10], Thompson proved a replacement theorem for abelian subgroups, showing that abelian subgroups could be replaced by *better* abelian subgroups, using abelian normal subgroups. Glauberman later proved a trickier replacement theorem involving subgroups of class two. These replacement theorems led to smarter choices that could prove stronger versions of normal complement theorems. The upshot: better control than ever before on the Sylow subgroups and p-structure of simple groups.

4.2. **Uniqueness subgroups and the maximal subgroup theorem.** As described earlier, p-local analysis studies the structure of nontrivial p-subgroups and their normalizers, which are called p-local subgroups. The structure of p-local subgroups can get messy, and one of the many questions that may arise is whether the collection of p-local subgroups has a largest element. The existence of a largest element would make the study of questions of conjugacy easier to handle.

One result in this direction is the maximal subgroup theorem due to Thompson (see [3, Theorem 8.6.3, Page 295–298],for instance). This gives condition under which certain subsets of the set of normalizers of p-subgroups in a given p-Sylow subgroup have maximal elements. When such a maximal element exists, it is termed a **uniqueness subgroup**.

References

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[2] *A brief history of the classification of the finite simple groups* by Ronald M. Solomon, *Bulletin of the American Mathematical Society*, ISSN 10889485 (electronic), ISSN 02730979 (print), Volume 38, Page 315–352 (Year 2001): An expository paper by Ronald Mark Solomon describing the 110-year history of the classification of finite simple groups.

[3] *Finite Groups* by Daniel Gorenstein, *American Mathematical Society*, ISBN–10 0821843427, ISBN–13 9780821843420

[5] Alas, out-of-the-box thinking ceases to be out-of-the-box pretty quickly. Thompson's methods were widely used by other group theorists.

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