FUNCTION SPACES – AND HOW THEY RELATE

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Abstract.

1. Function spaces

1.1. Why functions? This section is more about the gameplan and general philosophy that we'll follow. The basic premise (which many may find disagreeable) is the following:

> We are interested in the functions of certain types from certain kinds of spaces, to $\mathbb{R}$ or $\mathbb{C}$.

We're not interested in a function here or a function there. We're interested in the collection of all functions to $\mathbb{R}$ or to $\mathbb{C}$.

Why are we interested in these? Functions describe lots of things. For example, a function on a "physical body" could be used to describe the temperature at every point on the body.

Functions to $\mathbb{R}$ or $\mathbb{C}$ are *scalar-valued*: they have their image in a field. We are often interested in vector-valued functions, for instance, vector fields on open sets in Euclidean space. However, the theory of vector-valued functions, in the finite-dimensional case, isn't very different from that of scalar-valued functions, because, after all, the vector-valued function can be described by its scalar components. This raises a number of interesting points that we shall explore as we proceed.

Another important point is that for most practical purposes, theorems which work for functions to $\mathbb{R}$ also work for functions to $\mathbb{C}$. The main differences are:

1. For functions to $\mathbb{R}$ we use the absolute value to define the norm, and for functions to $\mathbb{C}$ we use the modulus of a complex number to define the norm.
2. The *inner product* we take for real vector spaces is a symmetric bilinear positive-definite form, whereas for complex vector spaces, we use a Hermitian inner product.

Any real function space can be embedded into the corresponding complex function space, and the norm (and inner product, if defined) for the real case, are simply the restriction to the real case of the norm and inner product defined for the complex case.

1.2. What space? What is the "function space"? Loosely speaking, a **function space**(defined) is a subquotient of the vector space of all functions from some set X to $\mathbb{R}$ or $\mathbb{C}$. A subquotient here means a quotient of a subspace (or equivalently a subspace of a quotient space).

Two things are worth mentioning here:

- We go to the *subspace* because we don't want to consider all functions.
- We take a *quotient* because we want to consider functions upto equivalence. For instance, we may be taking equivalence with respect to a measure (this is the most common instance; in fact, when we are dealing with a measure space, it is the only instance of concern to us).

So what we are considering *are* functions, but they're not all functions, and they're not *honest specific functions*; they're equivalence classes of functions.

1.3. Function spaces are vector spaces. A function space is, first and foremost, a vector space:

- Any scalar times a function in the space is also in the space
- A sum of two elements in the space is also in the space

This means that before we start taking out the hammer of function spaces to a particular property of functions, we need to show the above two things. Obvious? Not always. Sometimes, showing that a sum of two things in the space is also in the space is a pretty hard task.

1.4. **Function spaces are more.** One of our typical recipes for constructing function spaces is as follows:

(1) Start out by defining a translation-invariant and $\mathbb{R}$-linear distance function on the space of *all* functions, that takes values in $[0, \infty]$.
(2) Prove that this distance function satisfies the triangle inequality.
(3) Quotient out by the elements at distance 0 from 0, and get a distance function on the quotient space.
(4) Take the connected component of the identity. This is a vector subspace, and is the set of those elements at finite distance from 0.

Sounds familiar? This is the way we obtain the L^p-spaces. In other words, the L^p-spaces come as the connected component of the identity in the p-distance topology. Different connected components are at distance ∞ from each other.

This highlights something about function spaces. Many a function space can be given a metric, and in fact this can be extended to a "metric" on the whole space; the distance between two points in the same coset is finite while the distance between two points in different cosets is infinite. The connected component of identity then becomes a "normed vector space".

This is in some sense a "natural choice of metric" on the function space. Studying the function space without this metric is like studying $\mathbb{R}$ as a topological space without a metric.

1.5. **Algebraic structure on a function space.** We know that the set of *all* functions from a set to a field is not just a vector space; it's an *algebra* over the field. So it's natural to ask the question: to what extent are the function spaces we are considering subalgebras? Most of them aren't, though a select few are. I'll clarify my meaning:

- A *subalgebra* is just understood to be a vector subspace that is also multiplicatively closed. It need not contain the constant function 1
- A *unital subalgebra* is understood to be a vector subspace containing the constant function 1
- Given a subalgebra A and a smaller subalgebra B we say B is an ideal in A if multiplying any element of B with any element of A gives an element of B.

2. Various function spaces

2.1. **On a topological space.** Suppose X is a topological space. Then, we can consider the following function spaces on X:

(1) **Continuous functions**: The space $C(X, \mathbb{R})$ of all continuous real-valued functions (resp. the space $C(X, \mathbb{C})$ of continuous, complex-valued functions). This is a subspace of the space of all functions. In this case, the vector space nature isn't too hard to prove.
$C(X, \mathbb{R})$ (resp. $C(X, \mathbb{C})$) is actually more than just a vector space: it's a unital subalgebra.
(2) **Compactly supported continuous functions**: The space $C_c(X, \mathbb{R})$ (resp. $C_c(X, \mathbb{C})$) of all continuous, real-valued functions with compact support. This is a subspace comprising those continuous functions that vanish outside a compact set. It is a vector space because a union of two compact subsets if compact.
$C_c(X, \mathbb{R})$ is a subalgebra but is unital if and only if the space X is compact (in which case it equals $C(X, \mathbb{R})$). Nonetheless, it is always true that $C_c(X, \mathbb{R})$ is an ideal in $C(X, \mathbb{R})$.
(3) **Continuous functions that taper at infinity**: The space $C_0(X, \mathbb{R})$ (resp. $C_0(X, \mathbb{C})$) of all continuous, real-valued (resp. complex-valued) functions that go to zero outside of compact subsets. This contains $C_c(X, \mathbb{R})$, it is a vector subspace, and it is a subalgebra. It is unital iff the space X is compact, in which case it is the whole of $C(X, \mathbb{R})$. It is also always an ideal.
(4) We can take the smallest unital subalgebras generated by $C_0(X, \mathbb{R})$ and $C_c(X, \mathbb{R})$ respectively. The former is the algebra of all functions that "converge at ∞" and the latter is the algebra of all functions that become constant outside a compact subset.

First question: is there a natural "norm", such that the elements with finite norm, are precisely the continuous functions? Not in general. The natural norm that we associate with continuous functions when restricting to a *compact* subset, is the L^∞-norm (the sup-norm). That's because on a compact space, continuous functions are bounded.

On the other hand, there are obviously bounded functions that are not continuous. So how do we characterize continuity? The idea is to take very small open sets around a point, i.e. the formal definition is:

f is continuous at x if for any ε, there exists a neighbourhood U of x such that $\|f - f(x)\|_\infty < \varepsilon$ restricted to U.

This isn't a single norm, though.

2.2. **On a metric space.** On a metric space, we can do a little better than talk of continuous functions. We can define the notion of Holder spaces.

Definition (Holder space)**.** For $\alpha \in [0, 1]$, the the $(0, \alpha)$-**Holder norm**(defined) of a function f is defined as:

$$\sup_{x,y \in X} \frac{|f(x) - f(y)|}{d(x, y)^\alpha}$$

The elements with finite norm are said to form the Holder space $C^{0,\alpha}$.

For $\alpha \in [0, 1]$, $C^{0,\alpha}$ is not just a vector space, it's also a subalgebra. If we allowed $\alpha > 1$ in the above definition, we'd still get a vector space, but it need not be a subalgebra.

The metric structure allows us to talk of the following notions:

(1) **Uniformly continuous functions**: We all know what uniformly continuous functions are. The uniformly continuous functions to $\mathbb{R}$, or to $\mathbb{C}$, form a unital subalgebra of the space of all continuous functions.

On a compact metric space, continuous is the same as uniformly continuous. We can of course talk of locally uniformly continuous, but for locally compact metric spaces, any continuous function is locally uniformly continuous.

(2) **Lipschitz functions**: These are functions in $C^{0,1}$. The Lipschitz property is fairly strong. Lipschitz functions form a unital subalgebra.

(3) $C^{0,\alpha}$ for $0 < \alpha < 1$: This is somewhere in between uniformly continuous and Lipschitz. In other words, the size of the algebra decreases as α increases. Uniformly continuous functions lie in all the $C^{0,\alpha}$, and all the $C^{0,\alpha}$ contain the algebra of Lipschitz functions.

(4) **Locally Lipschitz functions**: These are functions that are in $C^{0,1}$ locally. This is an important condition. As we shall see, for example, all C^1 functions on a differential manifold are locally Lipschitz.

2.3. **On a differential manifold.** This uses additional structure on X: the structure of a differential manifold.

(1) For every positive integer r, we have the algebra $C^r(X, \mathbb{R})$ or $C^r(X, \mathbb{C})$: this is the space of r times differentiable functions from X to $\mathbb{R}$ (or to $\mathbb{C}$). This is a unital algebra, and we have a descending chain of algebras:

$$C = C^0 \supset C^1 \supset C^2 \ldots$$

(2) $C^\infty(X, \mathbb{R})$ (resp. $C^\infty(X, \mathbb{C})$) is the intersection of all these subalgebras, and is again a unital subalgebra.

In order to put a norm and speak quantitatively, we need to fix a metric on the manifold. Differential manifolds do not come with a natural choice of metric; however, any differential manifold can be embedded inside Euclidean space, hence it can be given a metric. Regardless of what metric we give in this way, we can talk of notions like:

- Holder spaces
- Space of locally Lipschitz functions

2.4. **On a measure space.** This uses the structure of a measure space on X. The function space here is not honest specific functions; rather, it is functions upto equivalence with respect to the measure on X.

(1) The space M is the space of all functions (finite-valued functions) upto almost everywhere equivalence. By finite-valued, we mean that any representative is finite outside a set of measure zero. M is an algebra under pointwise addition and multiplication. All the vector spaces we discuss below, live as subspaces of M.

(2) For any p with $1 \leq p < \infty$: The space $L^p(X, \mathbb{R})$ (resp $L^p(X, \mathbb{C})$) is defined in the usual way. It is a normed vector space, where the norm is the p-norm for $p < \infty$.

The L^ps are *not* subalgebras in general (in fact, an L^p is a subalgebra for finite p iff there do not exist subsets of arbitrarily small measure). Moreover, the L^ps are unital only if X has finite measure.

(3) For $p < r$, we have the following: $L^p(X) \cap L^r(X)$ contains $L^s(X)$ for all $s \in [p, r]$. The intersections $L^p(X) \cap L^r(X)$ are important subspaces, as we shall see in the Riesz interpolation theorem.

We can also consider the intersection of all the L^ps for $1 \leq p < \infty$. An application of Holder's inequality yields that this intersection is a subalgebra; however, it is not unital unless X has finite measure.

(4) The space $L^\infty(X)$ is defined as the space of all essentially bounded functions (i.e. equivalence classes of functions with a bounded representative). L^∞ is a unital subalgebra.

When the measure space is finite, this lives inside each of the L^ps, and in particular in the intersection of the L^ps; in general, the inclusion doesn't hold either way.

(5) We also have notions of weak L^p. Weak L^1 is the space of those functions that satisfy a Markov-like condition. One can similarly define weak L^p for finite p.

Some points need to be noted. The space L^∞ doesn't require us to have a measure space; all we need is the notion of a subset of measure zero and a notion of what it means for a function to be measurable. Hence L^∞ makes sense, for instance, on any differential manifold. But the other L^p-spaces require a precise notion of measure, particularly in the non-compact case.

2.5. **Visualizing the L^ps.** The L^p-spaces have myriad relationshisp that can sometimes be confusing to remember. There is, however, a physical picture that can help one understand the relation between the L^ps. The picure encodes the full Venn diagram, and is also useful to visualize later results like the Riesz- Thorin interpolation.

Prescription for drawing the picture:

- Make a square in the Euclidean plane with vertices $(0, 1), (1, 0), (0, -1), (-1, 0)$. In other words, the square is the solution to $|x| + |y| = 1$.
- For any L^p, think of the L^p-space as the rectangle whose vertices are given by $(\pm 1/p, \pm 1 - 1/p)$.

Observe that all the L^p-spaces live inside the square, that no L^p-space is contianed in the other. The space L^∞ is the vertical line and the space L^1 is the horizontal line. The space L^2 is a square, which indicates self-duality. The spaces L^p and L^q are congruent rectangles: we can get one from the other geometrically by reflecting in the line $y = x$, which indicate that they are dual. Finally, the intersection of L^p and L^r is contained in all the L^s for $p \leq s \leq r$, a fact again reflected in the Venn diagram.

The key idea in this diagram is to scale the interval $[1, \infty]$ according to the reciprocals.

2.6. **Sometimes these things can be put together.** The spaces on which we are considering functions are usually everything: they're topological, they're smooth, and they have measures. Thus we can consider all the above function spaces. We want to relate these function spaces.

First, note that there's a fundamental way in which the continuous/smooth ones differ from the measure theory ones; the former are honest specific functions, and the latter are functions upto equivalence.

Second, note that it's hard to say *anything* without having some way in which the measure respects the topology or the differential structure. So let's describe the relevant compatibility assumptions:

- For topological spaces with a measure, we usually require the measure to be a *Borel measure*: it should be defined on all the Borel subsets. We also often require the measure to be *inner regular*. This means that the measure of any open subset is obtained by taking the supremum of the measures of compact subsets contained inside it.
- For smooth manifolds with a measure, we require that the measure should be Borel, and that it should be compatible with the smooth structure, i.e. have the same notion of measure zero subsets.

2.7. **Domains in Euclidean space.** The standard (and most frequent) example to bear in mind is $\mathbb{R}^n$, or a domain Ω (i.e. a connected open subset) in $\mathbb{R}^n$. Note that domains in $\mathbb{R}^n$ have some more function spaces associated to them:

(1) **Polynomial functions**: A polynomial function on a domain is the restriction to the domain of a polynomial function on $\mathbb{R}^n$. Since a domain has infinitely many points, different polynomials on $\mathbb{R}^n$ give different functions on the domain.

(2) **Trigonometric polynomial functions**: These are functions that are restrictions of trigonometric polynomials in scalar multiples of the coordinates. Note that under a linear change of variables, a trigonometric polynomial is still a trigonometric polynomial, because of the rules for sine and cosine of a sum.
(3) **Schwarz functions**: $\mathcal{S}(X, \mathbb{R})$ (resp. $\mathcal{S}(X, \mathbb{C})$) is a subspace of C^∞ comprising those functions f that are **Schwarz**(defined): for any polynomial p, and any iterated partial derivative g of f, $gp \in C_0$ (i.e. it goes to zero outside compact subsets).
   In particular, f itself is in C_0. Thus $\mathcal{S}(X)$ is a subalgebra but is not unital; in fact, it is an ideal in $C^\infty(X)$.

We can relate the existing function spaces for domains:

(1) *No two continuous functions are equivalent*: If f and g are two continuous functions on Ω, and $f = g$ almost everywhere, then $f = g$. This is because the value of a continuous function is determined by knowing the values of the function at any sequence of points sufficiently close to it.
   Thus, for any subspace of the space of continuous functions, quotienting out by the equivalence relation of "upto measure zero" doesn't have any effect.
(2) We can introduce some new spaces called L^p_{loc}: A function f is in L^p_{loc} if for every open subset U contained in Ω such that U is relatively compact (i.e. the closure of U is compact), the restriction of f to U is in $L^p(U)$.
   It is easy to see that L^p_{loc} contains L^s_{loc} for $p \leq s$. The largest space among these, is the space L^1_{loc}, and the smallest is the space L^∞_{loc} of locally bounded functions.
   Continuous functions are in L^∞_{loc}, and hence in all the L^p_{loc}, but are not necessarily in L^∞. In fact, a continuous function need not be in L^p for any p.
(3) The intersection of the space of continuous functions with L^∞ is the space of *bounded continuous functions*, denoted B or BC. The intersection with L^1 is the space of integrable continuous functions.
(4) Living somewhere inside the space of bounded continuous functions is the space of continuous functions that converge at infinity. This, as we may recall, is the unitization of the space C_0 of functions that approach 0 at ∞.
(5) The Schwarz functions live inside the intersection of all the L^ps. Moreover, it is also true that if f is a Schwarz function, and g is in L^p for some p, then fg is in L^r for all $r \geq p$.

2.8. **A little more on Holder spaces.** For convenience, we here develop the theory of Holder spaces only on domains in $\mathbb{R}^n$. The idea is to combine the notion of differentiation, with the notion of a Holder space $C^{0,\alpha}$.

Recall that, a little while ago, we had observed that there's no easy choice of norm under which the only elements of finite norm are the continuous function. However, the L^∞ norm is a good one because, *locally* at least, continuous functions are bounded. We now try to define some more norms which are to C^r, what the L^∞-norm is to continuous functions.

On a domain, the idea is to add up the L^∞-norm of the function, and all its mixed partials of order up to r. If the sum is finite, we say we're in the Holder space $C^{r,0}$. We can now add a term that plays a role analogous to the role of the supremum in the Holder space, to make sense of the spaces $C^{r,\alpha}$.

Some points to note: First, all the spaces here are subalgebras. Second, once we've got these spaces, we can *de-localize* them to get notions of "locally" Holder functions. The de-localization process is necessary, to, for instance...

What happens if we're working with differential manifolds instead of domains? The Holder norms don't make sense, but the notion of being locally Holder is still robust.

3. A crash course in normed vector spaces

3.1. **Normed vector spaces and Banach spaces.** The *normed vector space* idea is a way to forget that the function spaces we are dealing with actually arise as functions, and just treat them as elements of a vector space with a norm that gives the correct topology and the appropriate notions of closeness.

Here we give some definitions:

Definition. (1) A **topological vector space**(defined) is a vector space with a topology such that the addition and scalar multiplication operations are continuous with respect to the relevant product topologies.
(2) A **normed vector space**(defined) is a vector space equipped with a norm function, that commutes with scalar multiplication, which is zero only at zero, and which satisfies the triangle inequality. The induced metric associates to a pair of points the norm of their difference vector.
(3) A **Banach space**(defined) is a normed vector space that is complete with respect to the induced metric.
(4) An **inner product space**(defined) is a vector space along with an inner product. If we're on a real vector space, the inner product needs to be symmetric, bilinear and positive-definite. If we're on a complex vector space, the inner product needs to satisfy the conditions of a Hermitian inner product.
(5) A **Hilbert space**(defined) is an inner product space that is complete with respect to the metric induced by the inner product space.

There are analogous notions for algebras:

Definition. (1) A **topological algebra**(defined) is a $\mathbb{R}$-algebra or $\mathbb{C}$-algebra where the addition, multiplication, and scalar multiplication are all jointly continuous operations.
(2) A **normed algebra**(defined) is a topological algebra with a norm that is sub-multiplicative, i.e. in addition to the conditions for a normed vector space, the norm $\|\cdot\|$ must satisfy:

$$\|xy\| \leq \|x\| \, \|y\|$$

(3) A **Banach algebra**(defined) is a normed algebra that is complete with respect to the induced norm.

The "algebra" structure and the "inner product" structure are similar: an algebra structure is a $\mathbb{R}$-bilinear map from the algebra to itself, an inner product is a $\mathbb{R}$-bilinear map to the base field. They both give additional leverage to whatever we are studying.

The importance of completeness is that it allows us to say that Cauchy sequences are convergent, and we can find a point to which they converge *within* the space. However, completeness is not as powerful as might be suggested at first glance. The key limitation is that it doesn't guarantee convergence in all the possible ways we could imagine. For instance, if we're looking at $C[0, 1]$ in the uniform norm, then we can take functions in it that converge *pointwise* to the indicator function of a point, but do not converge uniformly.

How do the function spaces we've seen so far fit into these models?

(1) All the L^p spaces are Banach spaces. L^2 is a Hilbert space, and L^∞ is a Banach algebra.
(2) The space C_0 of continuous functions that go to 0 at ∞, is a Banach subalgebra of the algebra of all bounded functions. When the space also has a measure, it's a Banach subalgebra of the Banach algebra L^∞.
(3) The various C^rs are algebras; they aren't normed, but we can find subalgebras inside them that are Banach. How? Take the subalgebra $C^{r,0}$ of things with finite C^r-norm. This is a Banach algebra.
(4) The Schwarz space forms an algebra, and a very interesting one at that. It can be viewed as a dense subalgebra in practically all the L^ps for $p < \infty$, and is also a dense subalgebra of C_0.

3.2. **Some other ideas and conditions.** The **dual space**(defined) to a normed vector space is the space of bounded linear functionals on it. For any normed vector space, the dual space is a Banach space, and the main thing we're using here is that the *target space* of a linear functional, which is $\mathbb{C}$, is a Banach space.[1]

The dual space is only a vector space, it isn't an algebra. However, we can construct from it an algebra, by taking the algebra generated by this vector space. In other words, we can consider the algebra of

[1]The correct generalization is that linear operators from a normed vector space to a Banach space form a Banach space.

bounded polynomial functionals on a normed vector space. These are functionals that can be expressed as sums of products of bounded linear functionals. Any guesses as to why it's a Banach algebra?

Given any normed vector space, there is an injective norm-preserving embedding of the normed vector space in its double dual.

Definition. (1) A **reflexive space**(defined) is a normed vector space such that the natural map into its double dual is an isomorphism. In other words, any bounded linear functional on the dual space comes from the normed vector space itself.

Since any dual space is complete, a reflexive space must be complete. The converse is not necessarily true. Thus, reflexivity is completeness in a strong sense.

(2) Given a normed vector space X, we can define on it a topology of weak convergence. A sequence of elements $x_n \in X$ converge to a point $x \in X$ if for any bounded linear functional l on X, $l(x_n)$ approaches the point $l(x)$. In other words, *as far as bounded linear functionals can see*, $x_n \to x$.

(3) A normed vector space is termed **strictly convex**(defined) if any convex linear combination of points on the unit sphere, lies strictly in the interior of the unit ball. A somewhat more complicated version of this is the notion of a **uniformly convex space**(defined), where we demand a minimum amount of *roundness* i.e. we show that the norm of a convex combination is bounded away from 1 in a strong sense. Uniform convexity is sufficient to guarantee reflexivity.

(4) A normed vector space is termed **weakly complete**(defined) if whenever x_n is a weakly Cauchy sequence, x_n is weakly convergent. We say that x_n is weakly Cauchy if it is true that for any bounded linear functional l, the sequence of values $l(x_n)$ is convergent.

So how does this throw new light on the spaces we've been considering so far?

(1) For $1 < p < \infty$, the spaces L^p are uniformly convex, hence reflexive. The dual to L^p is L^q where p and q are Holder conjugates.
(2) For $p = 1$, L^p is complete, but *not* reflexive. It is also not strictly convex. A simple picture of this is L^1 of a two-element set, which is the set$|x| + |y| = 1$ in $\mathbb{R}^2$. This "sphere" is far from round.
(3) For $p = \infty$, L^p is complete, but *not* uniformly convex, in fact not even strictly convex. To see this, again take L^∞ of a two-element set. The sphere is the square $\max\{|x|, |y|\} = 1$: far from round.
(4) The space C_0 is not reflexive (hence it's not uniformly convex). That's because it's dual is L^1, and the dual of L^1 is L^∞, which is significantly bigger than C_0.

4. Linear operators

4.1. **What are linear operators?** We are interested in linear operators from one function space to another. There are, broadly speaking, three possibilities:

- The two function spaces are the same
- The two function spaces are on the same domain, but are different
- The two function spaces are on different domains

In addition to linear operators from one function space to another, we also consider bilinear operators. These are maps from a product of two function spaces to a third function space, that are linear in each. We'll study operators of the various kinds, as we proceed.

4.2. **Algebra of linear operators.** The linear operators from a function space F, to *itself*, form an algebra. Here addition is pointwise and multiplication is by composition. This algebra is in general noncommutative, and is an infinite-dimensional analogue of the algebra of matrices in n variables.

We'll be studying things in two fundamentally different ways here:

- An abstract study of what we can say about the algebra of linear operators on a normed vector space, a Banach space, an inner product space, or a Hilbert space
- A concrete look at the situations where operators arise, when we are working with function spaces.

4.3. **The Banach algebra of bounded linear operators.** Suppose X is a Banach space. We can define the following Banach algebra:

- The elements are *bounded* linear operators from X to X
- The addition of elements is pointwise, as is scalar multiplication
- The norm is the operator norm
- The multiplication of elements is by composition

This is a noncommutative, but unital. Is it complete? (I don't see any reason why it shouldn't be).

Since it is a Banach algebra, we can now forget about how it arose (as operators from a function space to itself) and study the theory of Banach algebras. For what we're going to say in the coming sections, it *does* make a significant difference whether we're working with real or complex base field. We'll stick to the complex situation, because the field of complex numbers is algebraically closed.

4.4. **A bit of pause.** We've done three successive levels of generalization:

- We started out with some space on which we were interested in functions. We decided to look at the function space instead.
- Then, we took the function space, and decided to *forget* that it actually comes as functions. So we think of the function space simply as a normed vector space, or normed algebra, or Banach algebra, or whatever.
- Then, we studied bounded linear operators from the Banach space to itself. This gave a Banach algebra. Now, we're tempted to forget that the Banach algebra actually arose as an operator algebra, and simply study it as an algebra.

4.5. **Notions of spectrum for the Banach algebra.** Suppose A is a *unital* Banach algebra over $\mathbb{C}$. We define:

Definition (Spectrum)**.** The **spectrum**(defined) of an element $a \in A$ is the set of $\lambda \in \mathbb{C}$ such that $a - \lambda$ is not invertible.

What can we say about why an element isn't invertible? There could be a lot of reasons: perhaps it just *happened* not to be invertible. However, let's try to look at it positively. What are the equations it *can* solve, that'd make it non-invertible? In other words, how can I exhibit something to convince you that $a - \lambda$ is not invertible?

The trick is to show that $a - \lambda$ is a zero divisor. In other words, we find a $b \neq 0$ such that $b(a - \lambda) = 0$ (making $a - \lambda$ a right zero divisor) or we find a $c \neq 0$ such that $(a - \lambda)c = 0$ (making $a - \lambda$ a left zero divisor). This isn't the entire spectrum, but it's a fair chunk of it.

What does this translate to, if we now think of A as bounded linear operators from a function space F to itself?

- The condition of being a left zero divisor is equivalent to saying that $a - \lambda$ isn't an injective map. That's because $a - \lambda$ is zero on the image of c, and $c \neq 0$ tells us that the image of c is nonzero. This is equivalent to saying that λ is an eigenvalue for a (for the reverse direction of implication, we use Hahn-Banach). The set of such λ is said to form the **point spectrum**(defined) of a.
- The condition of being a right zero divisor is equivalent to saying that the range of $a - \lambda$ is not dense. One direction of implication is tautological, while the other is an application of Hahn-Banach. The set of such λ, minus those which already lie in the point spectrum ,forms the **residual spectrum**(defined) of a.

4.6. **Dual spaces and adjoint operators.** Given a Banach space F, we have a dual space F'. We now have two Banach algebras: the bounded linear operators from F to itself, and the bounded linear operators from F' to itself. So we have two noncommutative rings floating around. What is the relation between these rings?

There is a natural injective anti-homomorphism (injectivity follows from Hahn-Banach):

$$A_F \to A_{F'}$$

This sends an element of A_F to the *adjoint* element in $A_{F'}$ (defined in the only way that makes sense). The map is an anti-homomorphism because the order of multiplication gets reversed. Thus, the left zero divisors in A_F become right zero divisors in $A_{F'}$ and the right zero divisors in $A_{F'}$ becomes left zero divisors in A_F. Simple! Back in the language of normed vector spaces:

> For $a \in A_F$, the point spectrum of a is contained in the union of the point spectrum and residual spectrum of its adjoint, and the residual spectrum of a is contained in the point spectrum of its adjoint.

When F is a *reflexive* space, the adjoint map is an anti-isomorphism, so in that case, the left zero divisors become right zero divisors and the right zero divisors become left zero divisors.

4.7. **We'll see more.** A little later, we shall see that there are some theorems about the spectrum of an element in a Banach space. We'll see that the spectrum may be a single point, or it may have uncountable many points, but it's always contained in a disc in the complex plane. We'll also see that the radius of this disc can be computed by looking at norms of powers of the element.

5. Some examples of operators

5.1. **Multiplication operators.** Multiplication operators are operators between function spaces on the same domain. To start with, if F denotes the space of *all* finite-valued functions on a set X, we have a bilinear map:

$$F \times F \to F$$

that sends a pair of functions to their pointwise product. Given a fixed $f \in F$, we are interested in questions like: what is the image of a particular subspace of F under multiplication by f?

Let's review some important definitions in terms of the language of multiplication operator:

- A subspace A of F is a subalgebra if the multiplication operator restricts to an operator $A \times A \to A$.
- A subspace B of a subspace A is termed an ideal in A if the multiplication operator restricts to an operator $B \times A \to B$.

Here, now, are some basic facts about multiplication operators:

(1) *Holder's inequality* yields that for $a, b \in [0, 1]$, the multiplication operator restricts to a map:

$$L^{1/a} \times L^{1/b} \to L^{1/(a+b)}$$

In particular, we have a multiplication operator from $L^\infty \times L^p$ to L^p, and a multiplication operator from $L^p \times L^q$ to L^1 (where p and q are Holder conjugates).

(2) Common sense tells us that functions with compact support form an ideal inside the space of all functions (Because 0 multiplied with anything nonzero is still zero). Thus, continuous functions with compact support form an ideal in the space of all continuous functions.

5.2. **Differential operators.** Differential operators really make sense, not on any old function space, but on an *algebra* .So one can make sense of a differential operator on L^∞, but not on L^1.

Given a $\mathbb{R}$-algebra A, and an A-module B, a map $d : A \to B$ is termed a **derivation**(defined) if it satisfies the following:

- d is $\mathbb{R}$-linear:
- d satisfies the Leibniz rule:

$$d(ab) = a(db) + (da)b$$

This is a fairly abstract definition, but it turns out that for a differential manifold M, the derivations from $C^\infty(M)$ to $C^\infty(M)$ are in precise correspondence with smooth vector fields on M. Moreover, the derivations from $C^r(M)$ to $C^{r-1}(M)$ are in precise correspondence with C^r vector fields on M.

We can now define a differential operator:

Definition (Differential operator)**.** The **algebra of differential operators**(defined) on a $\mathbb{R}$-algebra A is a subalgebra of the algebra of all linear operators from A to A, generated by left multiplication maps ($g \mapsto fg$) and the derivations. The algebra of differential operators comes with a natural filtration: the r^{th} filtered component is differential operators of order $\leq r$. These are differential operators that can be expressed as sums of composites of derivations and multiplications, where each summand has at most r derivations.

How nice are differential operators?

What we'd ideally like to say is that differential operators are *bounded* in some sense of the word, but this unfortunately isn't true unless we choose our topologies carefully. The topology that we need to choose is the one that we saw for domains in $\mathbb{R}^n$: add up the L^p norms for all the mixed partials.

This is an attractive definition, and with this definition, we can in fact see that any differential operator is bounded with respect to this norm. Essentially, this is because when we differentiate in a direction, we are taking a fixed linear combination of the partials, and since the norms of the partials are incorporated in the norm of the function, we're doing fine.

However, if we just naively tried to relate the L^p-norm of a function with its derivatives, we'd end up in a severe mess.

So some points about differential operators:

- Differential operators make use of the *algebra structure* (i.e. the multiplication structure) in the function space.
- Differential operators make use of the smooth structure on the manifold.
- For differential operators to be bounded linear operators, we need to put norms on the function spaces that take into account the L^p-norms of the partial derivatives.

5.3. **Integral operators.** Integral operators make use only of a measure space structure, and do not involve any algebra structure on the function space.

Definition. Let X and Y be measure spaces. An **integral operator**(defined) with kernel function $K : X \times Y \to \mathbb{R}$ is defined as the map:

$$f \mapsto (y \mapsto \int K(x,y)f(x)\,dx)$$

Here, f is a function from X to $\mathbb{R}$, and the new function we get is from Y to $\mathbb{R}$.

The same kernel could be used to define an integral operator from functions on Y to functions on X.

We need to be a little careful with what we mean by an integral operator. In general, integral operators are just formulae; it is then upto us to make sense of what they honestly mean. To illustrate matters, let M be the space of all measurable functions, and A be a subspace of M. Let $\tilde{A}$ be the space of honest specific functions that becomes A when we go down to measure zero.

For any point $y \in Y$, the value:

$$\int K(x,y)f(x)\,dx$$

if it exists, is independent of the choice of representative in $\tilde{A}$, for a function class in A. This is the cool thing about integration: it forgets measure zero differences.

On the other hand, on the *range* side, there is a crucial difference. The naive, *pointwise* interpretation of an integral operator being defined for a function is:

For *every* $y \in Y$, the integral:

$$\int K(x,y)f(x)\,dx$$

makes sense and gives a finite value.

We could choose to weaken this somewhat, since we're upto measure zero:

For *almost every* $y \in Y$, the integral:

$$\int K(x,y)f(x)\,dx$$

makes sense and gives a finite value.

These are *pointwise operators* in an honest pointwise sense. So if A and B are subspaces of the space M of all measurable functions upto measure zero, and $\tilde{A}$ and $\tilde{B}$ are their lifts, then a pointwise operator from A to B almost lifts to a map from A to $\tilde{B}$. Thus, given an equivalence class of functions (an element of A) we obtain a *specific* pointwise defined element of $\tilde{B}$ (with a few points gone awry, yes, but still it's honestly defined almost everywhere).

This means that integral operators, when they make pointwise sense as above, transform equivalence classes of functions to honest specific functions. Why is this remarkable? Because given an equivalence class of functions only, it makes *no* sense to evaluate at a point. But given an honest specific function (even one that's ill-defined at a few points) one can make sense of defining it at most places.

However, it often happens that we can define integral operators from function spaces, that do *not* make sense at a pointwise level.

5.4. **Antiderivatives as integral operators.** An integral operator can be viewed as the integral with limits. Here are some situations:

Given a function $f \in L^1(\mathbb{R})$, we can consider the "antiderivative" of f as a map from $\mathbb{R}$ to $\mathbb{R}$, that happens to be 0 at 0. This is obtained by taking:

$$x \mapsto \int_0^x f(y)\,dy$$

This can be viewed as an integral operator, by the following method. Consider the function $K(x,y)$ which is 1 if $0 \le y \le x$, -1 if $x \le y \le 0$, and 0 elsewhere. Then, the above function is:

$$x \mapsto \int_{\mathbb{R}} K(x,y) f(y)\,dy$$

Thus, taking the antiderivative is an integral operator from $L^1(\mathbb{R})$ to $C^0(\mathbb{R})$ (actually to something better: locally absolutely continuous functions).

Here are some other "antiderivative"-like operators that we can think of as integral operators. Suppose f is a function defined on $\Omega \subset \mathbb{R}^n$. We want to define a new function that sends a nonnegative real R to the integral of f in the ball of radius R about the origin.

This can be viewed as an integral operator from $L^1(\Omega)$ to $C(\mathbb{R}_{\ge 0})$. where the kernel function $K(R,x)$ is 1 if x is in the ball of radius R, and 0 otherwise.

In fact, most of the integral operators that we see hide in some sense an antiderivative-like operation. The advantage of an integral operator is that we can define K in any manner whatsoever; it need not just takes values like 0, 1 or -1.

5.5. **Convolution product.** A convolution can be viewed as a special case of an integral operator, but it differs in two ways:

- It makes sense only for locally compact topological Abelian groups (that come with a compatible measure).
- It takes two inputs, and is bilinear.

If f, g are two functions on a locally compact topological Abelian group G, their convolution is defined as:

$$(f * g)(x) = \int_G f(y) g(x-y)\,dy$$

In other words, we are "adding up", for all ways of writing x as a sum of two things, f of the first thing and g of the second thing.

The convolution is commutative and associative, wherever it is defined.

Convolution is an example of a bi-integral operator: if we fix either function, we get an integral operator. For instance, if we fix g, we get an integral operator with kernel $K(x,y) = g(x-y)$, whereas if we fix f, we get an integral operator with kernel $K(x,y) = f(x-y)$ (the roles of x and y are reversed from the previous section). Thus, the same concerns that apply to integral operators apply here: we have the question of when the convolution makes pointwise sense, and when it makes sense even though it doesn't make pointwise sense.

6. A couple of general concerns

Before proceeding further, let's mention a few general concerns we have about operators. As we saw in the above examples, we don't usually define an operator by writing some gigantic infinite-dimensional matrix. Rather, we write some formula, and then argue that the formula makes sense as an operator between the required spaces.

As we already saw, there are two kinds of things we could do: have operators from a function space to itself, and have operators from one function space to another. The former comes with a nice algebra structure. The latter is also common.

A central question we shall repeatedly consider is:

> Let X, Y be measure spaces, and M, N be respectively the spaces of all measurable functions on X and on Y (upto equivalence). Suppose, further, that we have a formula that ostensibly takes a function on X and outputs a function on Y. How do we find out a function space F on X, and a function space G on Y, such that the formula defines a bounded linear operator from F to G (the *bounded* is with respect to whatever natural norm we are considering).

This is a vaguely worded question, but we attempt a partial answer in the next subsection.

6.1. **A quick look at the L^ps.** We begin with a simple lemma.

Lemma 1. Suppose A and B are normed vector spaces. Suppose C is a dense subspace of A and B is complete. Then, any bounded linear operator from C to B, extends to a bounded linear operator from A to B

Proof. Any point in A is the limit of a sequence of points in B. The sequence is Cauchy, and since the operator is bounded, its image is a Cauchy sequence. Hence, the image of the sequence is a convergent sequence in B (since B is complete). Moreover, the limit is independent of the particular sequence we chose, so we can set this to be the image of A. ☐

This suggests the following idea. Consider the space:

$$\bigcap_{1 \le p < \infty} L^p$$

We had observed earlier that this is a subalgebra, often without unit. Further, for a domain (in particular for $\mathbb{R}^n$), the Schwarz space is contained inside this, and is in fact dense in each of the L^ps.

Theorem 1 (Defining for Schwarz suffices)**.** If F is a linear operator from the Schwarz space to itself, and $\|F\|_{p,r} < \infty$, then F extends uniquely to a bounded linear operator from L^p to L^r, with the same norm. Here $\|F\|_{p,r}$ denotes the operator norm with the Schwarz space on the left given the L^p-norm and the Schwarz space on the right given the L^r-norm.

Moreover, if the original map was an isometry, so is the unique extension.

6.2. **Riesz-Thorin interpolation.** We first state an extremely powerful result, called the Riesz-Thorin interpolation theorem. We then look at a special case of this, called Schur's lemma, that can also be proved by a direct and clever argument.

Before beginning, recall the following fact:

> If $p \le s \le P$, then $L^s(X) \supset L^p(X) \cap L^P(X)$ and the latter is dense in the former.

The fact that the latter is dense in the former is attested to by the fact that the Schwarz functions, that are dense in all the L^ps for finite p, live inside the latter.

Now suppose F is an integral operator that we've somehow managed to define from L^p to L^r and from L^P to L^R, both the norms are bounded, and the definition agrees on the intersection. Then for any $s \in [p, P]$, and for $u \in [r, R]$ we want to know whether we get a map from L^s to L^u. In other words, we want to investigate what we can say about the boundedness of the map restricted to the intersection:

$$L^p \cap L^P \to L^r \cap L^R$$

If such a map has bounded (s, u)-norm, then by the density argument, it extends uniquely to a map from L^s to L^u.

The Riesz-Thorin interpolation theorem is a wonderful theorem that gives us sufficient conditions for this. The idea behind the theorem is simple: invert the exponents, and thus go from $[1, \infty]$ to $[0, 1]$.

There, take a convex linear combination of p and P, and take the *same* convex linear combination of r and R. Then you're guaranteed bounded operator norm.

Theorem 2 (Riesz-Thorin interpolation theorem)**.** Suppose F defines a bounded linear operator from L^p to L^r, and from L^P to L^R, such that the operators agree on the interaction. If $1/s = t/p+(1-t)/P$ and $1/u = t/r+(1-t)/R$, then F has finite (s,u)-norm, hence defines a bounded linear operator from L^s to L^u. Moreover, the bound on the operator norm (s,u) is given by operator norm from p to r^toperator norm from P to R^{1-t}.

We'll not sketch a proof here.

7. More on integral operators

7.1. **Composing integral operators.** Is a composite of integral operators integral? The answer is *yes*, if we do a formal computation with Fubini's theorem. Suppose X, Y and Z are measure spaces. Suppose $K : X \times Y \to \mathbb{R}$ is one kernel and $L : Y \times Z \to \mathbb{R}$ is another kernel. Let's try to compose the integral operators:

$$z \mapsto \int_Y L(y,z) \int_X K(x,y) f(x)\, dx\, dy$$

A rearrangement by Fubini shows that this is an integral operator whose kernel is:

$$(x,z) \mapsto \int_Y K(x,y) L(y,z)\, dy$$

Thus, at least at a formal level, a composite of integral operators is an integral operator.

Let's look at this in the special case of the antiderivative. Suppose our measure space is $[0,\infty]$ with Lebesgue measure, and we define the operator:

$$x \mapsto \int_0^x f(t)\, dt$$

The kernel of this operator is the map $K(t,x) = 1$ if $0 \le t \le x$ and 0 otherwise.

So what happens when we compose this operator with itself? A little thought reveals that the new kernel is:

$$K(t,x) = (x-t)^+$$

i.e. the positive part of the difference between x and t.

Thus, iterated integration within limits can be viewed as a *single* integral operators.

This tells us that if X is a measure space and A is a function space on X, we may be interested in the algebra of all integral operators from A to A. What can we say about these integral operators? Can we determine the pairs (p,r) such that the operator norm, viewed from L^p to L^r, is finite?

7.2. **Operators that are well-defined pointwise.** An integral operator can be thought of as a composite of two things: a multiplication operator, and the integration functional. Let's consider a situation where we're looking at an integral operator from a function space on X to a function space on Y, with kernel $K(x,y)$. If f is the input function, we want to ensure that for every $y \in Y$, the map:

$$x \mapsto K(x,y) f(x)$$

is in $L^1(X)$. By the preceding discussion, it suffices to say that this happens for *almost every* $y \in Y$. This brings us back to the multiplication operators situation, that we saw a little while ago: it's clear that if $f \in L^p$, then for almost every y, $x \mapsto K(x,y)$ is in L^q where q is the conjugate exponent to p.

Once we've ensured that the definition is valid pointwise, the next step is to ensure that the function we get at the end of it is again nice in some sense. Let's work out some examples to illustrate this.

Suppose we want to know when the integral operator by K lives in $Op_{1,1}$: in other words, when does it give a well-defined bounded linear operator from L^1 to L^1. The first step of the reasoning shows that for almost every y, $K(x,y) \in L^\infty(X)$. For the next step, we need to ensure that:

$$\int \left| \int K(x,y) f(x)\, dx \right| dy < \infty$$

A simplification and Fubini exchange shows that we need to ensure:

$$\sup_{x \in X} \int_Y K(x,y)\, dy < \infty$$

And that the value on the left side is precisely the norm as an operator from L^1 to L^1.

By a similar procedure, we can try determining the conditions under which we get an operator from L^1 to L^∞. In the case, the answer is simpler: we just get $K \in L^\infty(X \times Y)$. For L^∞ to L^1, we get, by a Fubini argument, that $K \in L^1(X \times Y)$, and for $L^\infty \to L^\infty$, we get the condition:

$$\sup_{y \in Y} \int_X K(x,y)\, dy < \infty$$

with the expression on the left side being the L^∞-norm (this is all part of a general phenomenon on adjoint operators that we shall see later).

It turns out that Riesz-Thorin interpolation, and the Schwarz route, that we mentioned in the last section, allows us to get around the problem of pointwise ill-definedness.

7.3. **Some non-pointwise definitions.** We now use the results of the previous section to give examples of situations where certain operators cannot be viewed as pointwise operators from L^p to L^r, but the Schwarz route allows us to view them in this way.

The primary example is the Fourier transform:

Definition (Fourier transform)**.** The **Fourier transform**(defined) of a function f on $\mathbb{R}^n$ is defined as the integral operator with kernel:

$$K(x,y) = \exp(-ix \cdot y)$$

It is an integral operator from function spaces on $\mathbb{R}^n$ to function spaces on $\mathbb{R}^n$. The image of f under the Fourier transform is denoted by $\hat{f}$.

The kernel of the Fourier transform is symmetric.

The kernel is in $L^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ (in fact, it has constant norm 1), which means that this has a good pointwise definition for $f \in L^1$. But there's no guarantee of a good pointwise definition for $f \in L^2$, or in any other L^p for that matter. We thus go the Schwarz route. Let $f \in \mathcal{S}(\mathbb{R}^n)$. We show the following:

- f maps the Schwarz space via a linear isomorphism to the Schwarz space.
- The map is a L^2-isometry, upto a normalization factor of 2π.

We can thus extend f uniquely to an isometry from L^2 to L^2. So the Fourier transform is an isometry from L^2 to L^2, and its behaviour on $L^1 \cap L^2$ is exactly the way we expected.

The Fourier transform, however, is dishonest and does not make pointwise sense, because for a particular $f \in L^2$, the formula may not make pointwise sense for any x.

We shall see that something similar happens for the Hilbert transform, that we'll encounter a little later in the text.

7.4. **Riesz-Thorin to the Fourier transform.** We've seen that the Fourier transform is an isometry from L^2 to L^2 (We did this by showing that it's an isometry from the Schwarz space to itself, and then extending). Let's observe one more basic fact: the Fourier transform is a bounded linear operator from L^1 to L^∞. This follows from a general fact about integral operators: if the kernel is in $L^\infty(X \times Y)$, we get a bounded linear operator from $L^1(X)$ to $L^\infty(Y)$.

We can now apply the Riesz-Thorin interpolation theorem, interpolating on the domain side between 1 and 2, and on the range side between ∞ and 2. When going to multiplicative inverses, we are trying to interpolate on the domain side between 1 and 1/2 and on the right side between 0 and 1/2. A little thought yields that we have a bounded linear operator from L^p to L^q where $1 \leq p \leq 2$ and q is the conjugate exponent to p.

A little word here. The only p for which we have a map that is genuine and pointwise is $p = 1$. In all other cases, we have a bounded linear operator on the Schwarz space in an honest pointwise sense, and we extend it to an operator on the whole of L^p.

The precise inequality here is termed **Plancherel's inequality**(defined), and the formulation of the inequality uses a more precise version of Riesz-Thorin which we haven't covered here.

7.5. **Typical application of Riesz-Thorin.** Riesz-Thorin interpolation is extremely useful, and we shall see it turn up repeatedly in the coming sections.

The typical strategy will be:

- Prove boundedness for certain special pairs (p, r). For this, use basic results like Holder's inequality, or facts about L^1 and L^∞ or just a quick Fubini argument.
- Use Riesz-Thorin to fill in the gaps.

7.6. **Riesz-Thorin for bilinear operators.** For bilinear operators, Riesz-Thorin can be applied using the *fix-one-at-a-time* idea. We shall see this in the next section, where we discuss convolutions.

8. More on convolutions

8.1. **The basic result.** For G a locally compact topological Abelian group, we had defined the convolution of two functions f, g on G as:

$$(f * g)(x) = \int_G f(x-y)g(y)\,dy$$

This is bilinear, and the first, probably surprising observation is that it is well-defined from $L^1 \times L^1$ to L^1. This is by no means obvious, because a product of L^1 functions is not necessarily L^1. However, the map actually makes sense *pointwise almost everywhere*, which means that although fg has no reason to be in L^1, it is true that for almost every x, the map:

$$y \mapsto f(x-y)g(y)$$

is in $L^1(G)$. In fact, we're saying something more: we're saying that the output function is *again* in $L^1(G)$. The only way I know of proving this is Fubini's theorem.

A quick note regarding Fubini's theorem may be worthwhile. The Fubini idea of interchanging two integrals is very useful, but it can also be abused a lot

8.2. **Riesz-Thorin to complete the convolution picture.** Common sense tells us that the convolution is well-defined from $L^1 \times L^\infty$ to L^∞. So, it's well-defined as a map $L^1 \times L^1 \to L^1$ and as a map $L^1 \times L^\infty \to L^\infty$.

Now fix $f \in L^1$, and consider map $g \mapsto f * g$. This is well-defined $L^1 \to L^1$ and $L^\infty \to L^\infty$, and bounded in both cases by $\|f\|_1$. Riesz-Thorin tells us that it is well-defined and bounded by $\|f\|$ as a map $L^p \to L^p$ for *every* p. Thus, we get that if $f \in L^1$ and $g \in L^p$, then:

$$\|f * g\|_p \le \|f\|_1 \|g\|_p$$

Now, fix $g \in L^p$. We use the fact that

9. Functoriality, products and other concerns

9.1. **Maps between underlying spaces.** We constructed function spaces by taking some set, or space, with additional structure, and looking at functions satisfying a certain property with respect to that additional structure. So one question is: do maps between the underlying spaces give maps between the function spaces? The obvious way of trying to get a map, of course, is *composing*, and that would be "contravariant" in nature.

Let's quickly review our various function spaces, and how the maps work out:

1. **Topological spaces**: If X and Y are topological spaces, and $f : X \to Y$ is continuous, composing with f defines a map from $C(Y)$ to $C(X)$. Moreover, if f is a proper map (inverse images of compact sets are compact) then $C_c(Y)$ gets mapped to $C_c(X)$, $C_0(Y)$ gets mapped to $C_0(X)$, and so on.
2. **Measure spaces**: Suppose X and Y are measure spaces, and a map $f : X \to Y$ has the property that the inverse image of any measurable subset is measurable, and the inverse image of any measure zero subset has measure zero. Then, we get an induced map from $M(Y)$ to $M(X)$, where these denote the spaces of measurable functions upto measure zero equivalence.

There are two very special cases of this: an open subset in a topological space, mapping by inclusion, with respect to a regular measure (for instance, domains in $\mathbb{R}^n$), and a product space projecting onto one of the factors.

(3) **Differential manifolds**: If X and Y are differential manifolds, and $f : X \to Y$ is a smooth map, then f induces a map from $C^\infty(Y)$ to $C^\infty(X)$ by pre-composition (and in fact it does something similar on every C^r). For it to preserve finiteness of norm, f should be a proper map.

9.2. **Product spaces: separate and joint properties.** Category-theoretically, the *product* of two objects is defined as something terminal with regard to maps *to* the object. In most of the cases we're interested in, the product of two spaces is just the Cartesian product, endowed with additional structure in the correct way. We now analyze how a function space corresponding to a product of two spaces, is related to the function spaces corresponding to the factors.

The general situation is like this. Given X and Y, and function spaces $F(X)$ and $F(Y)$, we have natural maps:

$$X \times Y \to X, \qquad X \times Y \to Y$$

This yields, by contravariance, maps of the form:

$$F(X) \to F(X \times Y), \qquad F(Y) \to F(X \times Y)$$

We could do many things with these two maps. For instance, we could define a *linear* map:

$$F(X) \times F(Y) \to F(X \times Y)$$

which takes a function of X, a function of Y, and then just adds them. We could also define a *bilinear* map:

$$F(X) \times F(Y) \to F(X \times Y)$$

which takes a function of X, a function of Y, and multiplies them. Combining both these ideas, we see that given functions on X and functions on Y, we can take sums of products of these. A function that's a product of a function of X and a function of Y is termed **multiplicatively separable**(defined), and we basically get the vector space spanned by multiplicatively separable functions.

Now the things we need to check are that in particular cases, we *do* still remain inside F when we multiply. For algebras of functions, this is clear, but it is *also* true in cases where the functions *do not* form an algebra; for instance, the function spaces L^p.

The second thing we study is the following non-unique and non-natural choices of maps. For fixed $y \in Y$, we get a map $X \to X \times Y$ by $x \mapsto (x, y)$. This should induce a backward map $F(X \times Y) \to F(X)$. By doing this for every y, we get a bunch of such maps, and similarly we can do this for every $x \in X$.

Now, we can ask the question: if a function on $X \times Y$ satisfies the property that its restriction to every X-fiber is in $F(X)$ and its restriction to every Y-fiber is in $F(Y)$, is the function in $F(X \times Y)$?

Not necessarily, as we shall see. There are functions that are fiber-wise continuous, but not continuous on the whole. The problem really lies in the ability to relate what is happening on different fibers. It's the fact that a network of vertical and horizontal roads cannot capture all possible directions of approach.

A function whose restriction to each fiber is in the function space for that fiber, is sometimes said to *separately* be that sort of function.

9.3. **Products in each of the cases.** Let's now review the various kinds of function spaces and see what can be said about products in each of these:

(1) **Topological space**: For a product of topological spaces, we do have natural embeddings $C(X) \to C(X \times Y)$ and $C(Y) \to C(X \times Y)$, and this gives a notion of a multiplicatively separable function on $X \times Y$. By certain facts like Weierstrass approximation, we can show that for spaces like $\mathbb{R}^n$, the span of multiplicatively separable functions is the whole space.

On a converse note, it is *not* true that any function that is separately continuous (i.e. continuous in each fiber) is jointly continuous.

We need to be more careful when dealing with continuous functions of compact support. If Y is not compact, the image of $C_c(X)$ in $C(X \times Y)$ does not land inside $C_c(X \times Y)$. However, a *product* of a function in $C_c(X)$ and a function in $C_c(Y)$ does land inside $C_c(X \times Y)$. Similar observations hold for C_0.

(2) **Measure space**: For a product of measure spaces, we do have natural membeddings $L^p(X) \to L^p(X \times Y)$ and $L^p(Y) \to L^p(X \times Y)$. Moreover, a product of images of these, is in $L^p(X \times Y)$. This is a kind of application of Fubini's theorem, if you want: the fact that when integrating a multiplicatively separable function, we can integrate the components against each of the measure spaces. Note that it is *not true in general* that a product of functions in L^p is in L^p, so we are really using that the two functions that we have are in some sense *independent*.[2]

   We can use a Weierstrass-approximation type argument again to show that the span of multiplicatively separable functions is dense.

   It's again not true that a function that is fiber-wise in L^p, must be in L^p.

(3) **Domains in Euclidean space**: We can here talk of the Schwarz spaces. As with C_c, it is *not* true that the image of $\mathcal{S}(X)$ in $C(X \times Y)$ is a Schwarz function. However, the product of an element of $\mathcal{S}(X)$ and $\mathcal{S}(Y)$, *does* land inside $\mathcal{S}(X \times Y)$.

   Again, a converse of sorts isn't true: a function could be *separately* Schwarz, and yet need not be Schwarz.

9.4. **Integrate to get functions on factor spaces.** The fiber-wise concept can be used to get one, somewhat more, useful concept. We observed that it wasn't enough to just put a separate condition on each fiber because that didn't guarantee any uniformity. However, we can conceivably do this:

- Assume a certain condition on each fiber, which guarantees that some norm is finite
- Now consider the function that sends $y \in Y$ to the norm at that fiber. Impose some good conditions on *this* function.

This is an *iterated integration* idea: we impose a condition on each fiber so that we get a value by integrating against each fiber (this is the *inner integration* step). Then we impose a condition on the new function that we got. This is the *outer integration step*. That this iterated integration procedure has exactly the same effect as working directly in the product space, is a *Fubini-type* result.

Let's analyze the applicability of this idea to all the function spaces we've been considering:

(1) **Measure spaces**: A function is in $L^\infty(X \times Y)$, iff, for almost every y, it is in $L^\infty(X)$, and the function that sends y to the L^∞-norm of the corresponding function, is in $L^\infty(Y)$.

   The same holds for L^p, for finite p. For finite p, it is an application of Fubini's theorem.

(2) **Schwarz space**: A function in $C^\infty(X \times Y)$ is in $\mathcal{S}(X \times Y)$ if for every fiber, it is Schwarz, and the rate of decline on the fibers is uniform.

10. Getting bigger than function spaces

Is the space M of measurable functions, the biggest possible space? Not necessarily. We can conceive of, and work with, *bigger* spaces. To understand these bigger spaces, first observe that there is the famous inner product:

$$\langle f , g \rangle = \int f\overline{g} \, dm$$

(For real-valued functions, we do not need complex conjugation).

Now, this does *not* define an inner product on all of M. In fact, we know that given any function in L^p, the functions against which its inner product is finite and well-defined, and precisely the functions in L^q where q is the conjugat exponent of L^p. The Riesz representation theorem tells us that for finite p, L^q are precisely the linear functionals on L^p.

10.1. **A Galois correspondence.** A nice way of looking at dual spaces is to consider the Galois correspondence picture. Let X be a measure space, and M be the space of all measurable functions on X, upto measure zero. Define the following binary relation on M as:

> f is related to g iff the inner product of f and g is well-defined and finite i.e. iff $f\overline{g}$ is in $L^1(X)$.

This is a binary relation on the set M; hence we can use it to define a Galois correspondence (this is in a similar way as we define a Galois correspondence between a ring and its spectrum). The correspondence is as follows: it sends a subset S of M to the set of all functions $f \in M$ that are related to *every* element of S.

[2] The probability-theoretic interpretation is that a random variables with finite expectation need not have finite expectation; however, a product of *independent* random variables with finite expectation has finite expectation

What does this Galois correspondence do? First, observe that we might as well assume that S is a vector space, because the elements related to S are the same as the elements related to the linear span of S. The Galois correspondence then sends S to those elements of M that give rise to elements of S^* (in a purely linear algebraic sense).

Some cautions and observations:

- The map we have defined (we'll call it the dual inside M), takes a vector subspace S of M and sends it to a vector space comprising those functions that induce elements in the algebraic dual space to S. However, it does *not* include all elements of the algebraic dual of S.
- We haven't yet put a topology on S, so we cannot talk of the notion of bounded linear functionals, as yet.
- The abstract nonsense of Galois correspondences tells us that applying this map *thrice* has the same effect as applying it *once*.[3] Thus, there are certain subspaces with the property that they equal their "double duals" inside M. This is *not* the same as saying that the vector space equals its algebraic double dual.
- It turns out that if we look at L^p just as a vector space (without thinking of the p-norm) and take its dual inside M, we'll get precisely L^q, where q is the conjugate exponent to p. We can then go backwards and see that for finite p, the "correct" topology to put on L^p, under which we'd get all bounded linear functionals, would be the p-norm topology. For $p = \infty$, we see that putting the L^∞-norm ensures that the dual within M consists only of elements from L^1, but there are elements in the dual space that do not come from within M.
- This also indicates that there is something "natural" about the topologies on most of the subspaces of M that we are considering. In other words, starting with a subspace S, we first find its dual inside M, and then topologize it so that its dual inside M are precisely the bounded linear functionals on it inside M.

10.2. **Going beyond functions.** We've now set up a sufficiently huge language to see one natural way of generalizing functions. Namely, instead of trying to take duals inside M, we try to take the whole dual space, as bounded linear functionals under the relevant topology. Unfortunately, there is no *single* space in which we can study everything.

Some of the examples of this are:

- **The space of all possible finite real/complex measures**: The elements of this space are finite real/complex measures, and we add the measures in the obvious way. Where does the space of real/complex measures live? For one, any such measure defines a linear functional on the space of bounded functions. We need to be careful here: we're talking of functions that are honestly bounded, rather than equivalence classes of bounded functions (because that'd be with respect to another measure).

  In particular, the space of all possible finite real/complex measures lives inside the dual space to BC, the space of bounded continuous functions.
- The Lebesgue-Radon-Nikodym theorem now makes more sense. It says that if (X, m) is a measure space and μ is any complex measure on X, then μ can be broken up as the sum of a measure that's absolutely continuous with respect to m, and a measure that's *singular* with respect to m: in other words, the two measures are supported on disjoint sets.

  The absolutely continuous part is the part that comes from within L^1, and the singular part is the part that comes from outside.

The upshot is that there are spaces bigger than function spaces: namely, spaces of measures. And some of these are totally orthogonal to function spaces, such as the singular measures.

There are certain special kinds of singular measures that come up frequently: the *discrete* singular measures. Discrete singular measures are measures that are concentrated on a discrete set. Note that singular measures could be far from discrete: we could, for instance, take an uncountable measure zero set, and provide a measure on it using a bijection with Euclidean space. That'd be singular but would be orthogonal to any discrete singular measure.

Perhaps the most important discrete singular measure, from which the others are built, is the delta measure (also called the delta distribution):

[3] The same logic as we use for centralizers in rings, and for the correspondence between a ring and its spectrum.

Definition (Delta measure)**.** The delta measure on a set X, for a point $x \in X$, is the measure that assigns measure 1 to the singleton subset $\{x\}$, and measure zero to any subset not containing $\{x\}$.

Integrating against δ_x is akin to evaluating at x.

11. Differential equations

11.1. **A little pause.** Where have we been going? Our basic assumption is that it's interesting to study functions from a given set or space, to the reals or complex numbers. Starting with these vector spaces of functions, we've built a theory that surrounds them: we've gone on to study operators between function spaces, spectra of operators, and some particular examples. Now it's time to justify our investment.

Why study different function spaces?

We're looking for a certain, ever-elusive, function that satisfies some conditions. If we could make do with a nice and good C^∞-function, we shouldn't bother with all the measure theory stuff. On the other hand, we may not always be able to get the C^∞ functions. We need to calibrate just how good or bad our functions can get, and look in increasing levels of generality.

Why not then just look at the largest possible function spaces?

That's bad, because then we cannot use the specific machinery that is applicable to the smaller, nicer, more well-behaved function spaces. The Occam's razor cuts backwards: to solve the differential equations as simply as possible, we need to have a theory of as many different kinds of function spaces, as we can get our hands on.

Why all this structure of norms, operators etc.? Where do they come in?

That's what we'll see in this section.

One thing we'll see in the coming sections in a return to the "real"-valued functions, as opposed to the complex-valued functions. Since the reals form a subfield of the complex numbers, "real inputs give real outputs", which means that if everything in the differential equation is real, one can expect to come up with real solutions.

11.2. **How do differential equations arise?** Differential equations arise from instantaneous laws. These could be physical laws, social laws; all kinds of laws. The idea is: "the way something is changing, depends on the way it is". For instance, a differential equation like:

$$\frac{dx}{dt} = x$$

reflects the law that the rate at which x changes with respect to times, equals the value of x. The more it is, the faster it changes.

The general setup for a partial differential equation is:

- There are some *independent variables*: These are the spatial or time variables in which the system is embedded. For instance, if a particle is moving through time, the independent variable is time. If we are tracing the temperature in the universe through space and time, the independent variables are space and time. (the variables are assumed to be real-valued).
- There are some *dependent variables*: These are variables that depend, in some sense, on the independent variables. (the variables are again assumed to be real-valued)
- There is a *system of differential equations*: Each equation in this system relates the independent variables, dependent variables, and *partial derivatives* of the dependent variables in the independent variables.

Formally, there is a "configuration space" for the independent variables (the set of all possible value combinations), and a configuration space for the dependent variables, and there's a map from the former to the latter. What the differential equation does is pose constraints on how this map must behave.

11.3. **Order, degree and linearity.** Differential equations could look real funny. They could involve partial derivatives, mixed partials, they could involve functions like sin and cos applied to the derivatives, they could involve just about anything. Where, then, does all the beautiful theory that we've constructed for normed vector spaces, fit in?

It fits in because the differential equations that we'll be considering, for the initial part, have a very special structure. Here are some definitions:

Definition. (1) A (nonlinear) differential operator of order $\leq r$ is a map from the space of C^r functions to the space of C^0 functions, that involves a C^r function of the variable, the value of the function at the point, and all its mixed partials of order upto r.

(2) A (nonlinear) differential equation of order $\leq r$ is an equation of the form $F(f) \equiv 0$ where F is a nonlinear differential operator of order $\leq r$, and f is the function for which we need to solve.

(3) A **polynomial differential operator**(defined) of order $\leq r$ is a (nonlinear) differential operator that is a polynomial function of the variable, the value of the function at the point, and all the mixed partials. Similarly we have the notion of a **linear differential operator**(defined): this coincides with the notion of differential operator we saw earlier.

(4) A differential operator is said to have **degree**(defined) d if it is expressed as a degree d homogeneous polynomial of the mixed partials of order exactly r, plus a differential operator of order $\leq r-1$. In the situation where there is exactly one dependent variable and exactly one independent variable, this is the same as saying that the d^{th} power of the r^{th} derivative is expressed in terms of all the derivatives of lower order.

Note that *linear* differential operators are always first-degree, so in fact studying first-degree differential operators is a bit more general than studying linear differential operators. Indeed, a number of instantaneous laws are, by nature, first-degree. These laws somehow describe the highest derivative, in terms of the lowest derivative. For instance:

- A law may describe the rate at which water flows through a pipe, as a function of a number of *degree zero* quantities.
- A law may describe the rate at which a particle accelerates, in terms of its speed, its position, and other factors. Here, it's a first-degree second-order differential equation, because the second derivative with respect to time is described using the derivatives of lower orders.

11.4. **Linear differential operators.** The study of linear differential operators differs fundamentally from the study of nonlinear ones. For linear differential operators, we can apply the entire theory of Banach algebras between function spaces. The kernel and image of linear differential operators are both vector subspaces. If L is a linear differential operator, then the solution set to $Lf = g$ can be obtained by finding a *particular solution* and then finding all solutions to $Lf = 0$ (the *kernel* of the differential operator) and adding up.

Thus, if L is a linear differential operator, the question of whether $Lf = g$ has a solution breaks down into two parts:

- Finding the *kernel* of the map
- Finding the *image* of the map. More specifically, knowing whether g is the image of something, and if so, what the inverse image looks like

12. More function spaces:Sobolev et al.

12.1. **Bigger function spaces, more solutions.** This is based on the central theme: *can we make sense of differentiation for more functions than we're willing to admit?* The idea behind doing this is to provide an alternative definition of derivative, with the property that for C^1 functions, this coincides *exactly* with the usual notion of derivative.

One such criterion is measure-theoretic. It goes back to Stokes' theorem, which states that if you integrate the gradient of a function over a domain, it is equivalent to integrating the function over the *boundary* of the domain. More generally, one could *weight* both integrals by a test function. We can combine this with the product rule:

Let Ω be a domain in $\mathbb{R}^n$ and $u \in C_c^2(\Omega)$. Then, for any function $f \in C^2(\Omega)$, we have that:

$$\int_\Omega (\nabla f \cdot \nabla u + u\Delta f)\, dm = \int_\Omega \nabla(u\nabla f)\, dm = 0$$

The last follows because we can take the boundary of the support of Ω. The integral of $u\nabla f$ on the boundary equals the integral of its derivative inside, so we get zero.

In other words, if $\langle\ f\ ,\ g\ \rangle$ denotes the integral of the product fg over Ω, we get:

$$\langle\ \nabla f\ ,\ \nabla u\ \rangle = -\langle\ u\ ,\ \Delta f\ \rangle \tag{1}$$

A similar reasoning shows that:

$$\langle \, \Delta u \, , \, f \, \rangle = - \langle \, \nabla f \, , \, \nabla u \, \rangle \tag{2}$$

Combining, we get:

$$\langle \, u \, , \, \Delta f \, \rangle = \langle \, \Delta u \, , \, f \, \rangle \tag{3}$$

All these equalities hold, at least *a priori*, only under the assumption that u is continuous with compact support.

Now, this suggests *two* alternative definitions of a twice-differentiable function:

(1) The definition compatible with equation 2, states that there is a vector-valued function α, such that for any $u \in C_c^2(\Omega)$:

$$\langle \, \Delta u \, , \, f \, \rangle = - \langle \, \alpha \, , \, \nabla u \, \rangle$$

(2) The definition compatible with equation 3, states there there is a function $g \in L^1_{loc}(\Omega)$, such that for any $u \in C_c^2(\Omega)$:

$$\langle \, u \, , \, g \, \rangle = \langle \, \Delta u \, , \, f \, \rangle$$

In other words, we can find a function to play the role of the Laplacian.

We can thus define various *new* kinds of function spaces, as those function spaces for which such "derivatives" exist. Point (1) gives functions that are, in a weak sense, once differentiable, and point (2) gives functions that are, in a weak sense, twice differentiable. The classes of functions here are somewhat bigger than all the functions in C^1 or C^2, yet they are smaller than the class of all functions in L^1_{loc}.

This leads to the notion of Sobolev spaces. Roughly, a Sobolev space is the space of functions which have "derivatives" in the sense of satisfying the integral equations with respect to any test function.

12.2. **Formal definitions of weak derivatives.** Let Ω be a domain in Euclidean space.

For convenience, we will say that u is a **test function**(defined) if $u \in C_c^\infty(\Omega)$.

Definition (Weak derivative)**.** Let α be a multi-index. Let $u, v \in L^1_{loc}(\Omega)$. Then, we say that v is the α^{th} **weak derivative**(defined) of u, or in symbols:

$$D^\alpha(u) = v$$

if for any test function φ, we have:

$$\int_\Omega u D^\alpha \varphi \, dm = (-1)^{|\alpha|} \int_\Omega v\varphi \, dm$$

The condition is designed to satisfy Stokes' theorem. So if the function actually *does* have an α^{th} partial in the strong sense, that must also equal the weak derivative.

Uniqueness follows because there is no function in L^1_{loc} whose inner product with *every* function in C_c^∞ is 0.

12.3. **Sobolev space.**

Definition (Space of k-times weakly differentiable functions)**.** A function in L^1_{loc} is termed k-times weakly differentiable if, for every multi-index α with $|\alpha| \leq k$, there exists an α^{th} weak derivative of the function.

We now define the Sobolev spaces:

Definition (Sobolev space $W^{k,p}$)**.** The **Sobolev space**(defined) $W^{k,p}(\Omega)$ is defined as the vector subspace of $L^1_{loc}(\Omega)$ comprising those functions such that for each multi-index α with $|\alpha| \le k$, the α^{th} weak derivative exists, and lives in $L^p(\Omega)$ (this includes the empty index, so in particular any function in $W^{k,p}$ is in L^p).

Here, k is a nonnegative (possibly infinite) integer and $p \in [1, \infty]$. Clearly, $W^{k,p}$ becomes smaller as k becomes larger, and $W^{0,p} = L^p$.

If we *fix* k, then the relative inclusion relationships between the $W^{k,p}$s for varying p, are similar to the relations between the L^ps. More precisely:

(1) The $W^{k,p}$s are normed vector spaces, where the norm is the p^{th} root of the sum of the integrals of p^{th} powers of all the mixed partials of order at most k. In symbols:

$$\|u\|_{W^{k,p}(\Omega)} = \left(\sum_{|\alpha| \le k} \int_\Omega |D^\alpha u|^p \ dm \right)^{1/p}$$

for finite p. For $p = \infty$, we take the sum of the essential suprema of all the derivatives.

Note that we *add the powers* before taking the p^{th} root for finite p, and for $p = \infty$, we add the essential suprema, rather than taking the essential supremum of the sum.

(2) In fact, the $W^{k,p}$s are Banach spaces, i.e. they are complete with the above norm.

(3) The space $W^{k,\infty}$ is a Banach algebra, and the space $W^{k,2}$, also denoted H^k, is a Hilbert space.

(4) For $s \in [p, r]$, the space $W^{k,s}$ contains the intersection $W^{k,p} \cap W^{k,r}$ (*is this really true? I'm guessing a Holder's on each of the mixed partials should do it*). Thus, for any fixed k, the Venn diagram described in section 2.5 transfers to the case of the $W^{k,p}$s.

(5) We also have notions of $W^{k,p}_{loc}$. Dowe have notions of weak $W^{k,p}$? No idea.

12.4. **Differential operators between the Sobolev spaces.** Recall that in section 5.2, we had said that *once* we have an algebra of functions, we can consider the differential operators on that algebra. The only example we had at that time was that algebra of C^∞ functions.

Now, with Sobolev spaces, we have a few more choices of algebra. Namely, the space $W^{k,\infty}$ is a Banach algebra for any k, and any of the weak first derivatives gives a differential operator from $W^{k,\infty}$ to $W^{k-1,\infty}$. And if we look at the space $W^{\infty,\infty}$, then we can define differential operators of any order on it.

This suggests another way of looking at Sobolev spaces: they are the largest spaces to which the differential operators that we can define for C^k (or C^∞) extend with range in the L^ps.

So a question arises: under what conditions are the spaces of honestly differentiable functions, dense in the Sobolev space? Under what conditions is the weak derivative just a linear algebra way of extending the notion of derivative we already have for the C^r spaces? We define:

Definition ($W^{k,p}_0$: closure of C^∞_C)**.** Let Ω be a domain in $\mathbb{R}^n$. Then the space $W^{k,p}_0(\Omega)$ is defined as the closure of $C^\infty_c(\Omega)$ in $W^{k,p}$, with respect to the $W^{k,p}$-norm. The space $W^{k,2}_0$ is often written as H^k_0.

In general, $W_0 \neq W$, which means that we cannot always approximate by compactly supported continuous functions. Loosely speaking W_0 is those elements where all the weak derivatives approach zero on the boundary. On the other hand, any element of W *can* be approximated by C^∞-functions.

12.5. **Approximation by smooth functions.** The first theorem states that if U is a *bounded* domain (so it is relatively compact), then any smooth functions are dense in $W^{k,p}$.

Theorem 3 (Global approximation by smooth functions)**.** Suppose U is a bounded domain, and u is a function in $W^{k,p}(U)$ for finite p. Then, there is a sequence of elements in $C^\infty(U) \cap W^{k,p}(U)$ that approaches u in the $W^{k,p}$-norm.

The sequence of functions that we choose may well have elements that blow up on the boundary. the obstacle to doing better is that the boundary may be ill-behaved. We have another theorem in this regard:

Theorem 4 (Global approximation by functions smooth upto the boundary)**.** Suppose U is a bounded connected open subset of $\mathbb{R}^n$, such that ∂U occurs as the image of a smooth manifold under a C^1 map. Then, any element of $W^{k,p}(U)$ is the limit of a convergent sequence of elements of $C^\infty(\overline{U})$, in the $W^{k,p}$-norm.

By $C^\infty(\overline{U})$, we mean functions that extend smoothly to the boundary. Such functions are automatically in $W^{k,p}(U)$.

We'll see a little more of the theory of Sobolev spaces as we move on. For now, it is time to return to the world of differential equations.

12.6. **How this affects solving equations.** Solving differential equations is, roughly speaking, a *reverse* process to differentiation. What the above does is to enlarge the space is which we look for solutions. Thus, the differential equation:

$$\frac{\partial^2 x}{(\partial t)^2} = f(t)$$

can now be made sense of and solved even when the right side isn't a C^0 function, because we're now looking for the left side is a Sobolev space rather than in the constrained space C^2.

13. Heat and flow

13.1. **Flow equations.** The general idea behind a flow equation is the following:

- We have a huge (possibly infinite-dimensional) vector space or manifold. This is the *function space*, or the space of possible configurations
- We have a vector field defined on this manifold. This describes the *instantanous law.*
- We need to find the integral curves of this vector field. This describes the *evolution with time.*

The configuration space here is usually *itself a function space.* The vector field is the *instantaneous law* by which the function changes with time.

Let's give an example: the famous heat equation. Consider a "body" Ω, a connected open subset in $\mathbb{R}^n$. Suppose $u : \Omega \to \mathbb{R}$ is a scalar-valued function that describes the temperature on the body. u is an element in some (to be determined) function space on Ω. What function space depends on how reasonable we expect things to be. For instance, if we assume that temperature variation is always continuous, we can think of u as living inside $C^0(\Omega)$. We may also insist that the temperature should be bounded, or that it should be integrable, and so on. In other words, the reality of the physical situation constrains us to a particular function space, say F, of possible temperature functions on Ω.

The temperature naturally evolves with time. In other words, as time passes, the temperature function changes. Moreover, the speed with which the temperature changes depends *only* on the current temperature. In other words, we can construct a *vector field* on the infinite-dimensional space F and the way the temperature evolves is the integral curves of the vector field.

Flow equation for a single scalar dependent variable u are thus of the form:

$$\frac{\partial u}{\partial t} = \text{Some function of } u,\ x, \text{ and mixed partials in } x$$

The right side is *independent* of time (time invariance of physical laws, or of vector fields). If we think of it in terms of time, we get a first-order, first-degree differential equation. However, there are also higher partials in space involved on the right side.

For the time being, we'll consider situations where the right side is a *linear differential operator* in terms of the spatial variables. Note that even nonlinear differential operators give vector fields, but with linear differential operators, the vector fields behave in a particularly nice way: the vector field itself has a linear dependence on the point.

13.2. **Solution operators.**

Definition (Solution operator)**.** Suppose F is a function space on a domain Ω, and L is a linear differential operator from F to another function space (say G) on Ω. Then, the **solution operator**(defined) for L, called $S(L)$ is defined as follows: for any fixed $t \in \mathbb{R}$, $S_t(L)$ sends u to where the integral curve to the associated vector field will be, at time t.

If a solution operator exists for all $0 \leq t < \varepsilon$, it exists for all $t \in [0, \infty)$. If a solution operator exitss for all $|t| < \varepsilon$, it exists for all real numbers. The solution operators form a **local one-parameter semigroup**(defined) of maps on the function space, and if they're globally defined, they form a one-parameter group.

Do solution operators exist? Are they unique? And most importantly, are solution operators *nice* in some sense? Do they live inside the algebra of integral operators? What is the dependence of a solution operator on time? Let's consider this, and related, questions.

13.3. **Fundamental solution.** Suppose we have the following differential equation for a function $u : \mathbb{R} \times \mathbb{R}^n \to R$ where the first input is thought of as time t and the second input is thought of as space x:

$$\frac{\partial u}{\partial t} = Lu$$

where L is a linear differential operator in the second input. A **fundamental solution**(defined) is a solution Φ to this differential equation defined for $t > 0$ such that:

- *Limit at origin*: As $t \to 0$, $\Phi(t, \_)$ approaches the Dirac delta-function for 0.
- *Smoothness*: $u \in C^\infty(\mathbb{R} \setminus 0 \times \mathbb{R}^n)$
- *Normalization*: For any time $t > 0$, $\int_{\mathbb{R}}^n \Phi(x, t)\, dx = 1$

Such a fundamental solution Φ enjoys the following properties:

(1) Suppose $u_0 : \mathbb{R}^n \to \mathbb{R}$ is the *initial value* of an unknown solution u to this equation (in other words, it is the value at time $t = 0$). Then, for any time t, define:

$$u(t, x) = \int_{\mathbb{R}}^n \Phi(t, x - y) u_0(y)\, dy$$

This is a solution of the initial value problem, if u_0 is a bounded continuous function.

(2) Since Φ is infinitely differentiable, the solution function we obtain this way is C^∞ except possibly at 0, in *both* variables.

13.4. **Some interpretation of this.** The existence of a fundamental solution for a differential equation of the form $\partial u / \partial t = Lu$ shows the following:

(1) Given any initial-value problem where the initial value is a bounded continuous function, the solution exists and is unique for positive time. Moreover, it is smooth for positive time.
(2) Let's interpret this in terms of the infinite-dimensional spaces involved. We see that the space BC^∞ dense in BC, but more than that, we see that there is a kind of "natural" choice for a way to approximate an element of BC by elements of BC^∞: use the solution to the differential equation.
(3) Physically, it means that if a temperature distribution on a body is bounded and continuous at any given instant of time, then at any *later* instant of time, it must become smooth. This is a kind of *infinite propagation* idea.

13.5. **Some equations and their fundamental solutions.**

14. Pseudo-differential operators and quantization

We now introduce some of the leading ideas of pseudo-differential operators. We'll begin by looking at a very simple setup, and gradually increase the complexity of our ideas.

For simplicity, let's consider the simplest case: functions from $\mathbb{R}$ to $\mathbb{R}$. We have the algebra $C^\infty(\mathbb{R}; \mathbb{R})$ of all functions on $\mathbb{R}$. An important subalgebra of this is the algebra of *polynomial* functions, which can be described as the polynomial ring $\mathbb{R}[x]$.

Now, the algebra of derivations of this polynomial ring is the Weyl algebra: it is the quotient of the free associative algebra in two variables x and ξ, by the Weyl relation. Here, x is thought of as acting by (left) multiplication, while ξ is thought of as acting as differentiation. Explicitly it is:

$$\mathbb{R}\langle x, y\rangle / \langle\langle \xi x - x\xi - 1\rangle\rangle$$

Every element of this algebra has a unique expression as a "polynomial" in ξ, with coefficents (written on the left) coming from $\mathbb{R}[x]$. However, the "coefficients" do not commute with ξ, so it matters that we're writing the coefficient on the left.

What happens when we consider the whole ring $C^\infty(\mathbb{R};\mathbb{R})$? The algebra of derivations is now an algebra that looks like a *polynomial* in ξ but could be *arbitrary* C^∞ functions in x. In other words, we can get a unique "polynomial" expression in ξ with coefficients (written on the left) in $C^\infty(\mathbb{R};\mathbb{R})$. In other words, differential operators have polynomial dependence on the derivative operation, but C^∞-dependence on the variable x.[4]

This suggests a question: can we somehow generalize the notion of differential operators so that the dependence *both* on x and on ξ is C^∞? In other words, can we take C^∞-functions of the operation d/dx? This seems a somewhat strange thing to hope for, but surprisingly, it *can* be achieved, that too in possibly more than one way.

14.1. **The setup for pseudo-differential operators.** When we say pseudo-differential operator, we'll mean a linear pseudo-differential operator. I don't know if there is a suitable generalized nonlinear notion of pseudo-differential operator.

Definition (Theory of pseudo-differential operators)**.** Suppose Ω is a domain in $\mathbb{R}^n$. Then, there is a natural embedding of $\mathbb{R}$-algebra generated by all the first partials (a polynomial algebra in n variables), into the algebra of differential operators on Ω. A **theory of pseudo-differential operators**(defined) is the following data:

(1) A bigger algebra containing the algebra of differential operators (called the algebra of linear pseudo-differential operators)
(2) An extension of the embedding of the polynomial algebra $\mathrm{Pol}(U)$ in the algebra of linear differential operators, to an embedding of the algebra $C^\infty(U)$ (or some large subspace thereof) in the algebra of linear pseudo-differential operators

A priori, it is not clear whether algebras of pseudo-differential operators exist.

Let's see what we need to do to construct a theory of pseudo-differential operators. We need to somehow invent a machine that takes a polynomial and outputs the differential operator obtained by thinking of the variables in the polynomial as first partials. Moreover, this machine should be sufficiently robust to take in things more complicated than polynomials: say things in $C^\infty(U)$, or $C_c^\infty(U)$. The machine is already there for us: it's the Fourier transform.

14.2. **The ordinary and Weyl transform.** The ordinary transform, which achieve one family of pseudo-differential operators, is described as follows. Consider $a \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ as the *function* that's C^∞ in both x and ξ, and that we plan to turn into a differential operator. Then, the **ordinary transform**(defined) by a is the following integral operator applied to the *Fourier transform* of the function:

$$(x, \xi) \mapsto \frac{1}{(2\pi)^n} e^{ix\cdot\xi} a(x, \varepsilon\xi)$$

In other words, we start with f, take its Fourier transform, then apply the above integral operator, and the final outcome that we get is what the differential operator corresponding to a, does to f.

Let's take some special cases, to get a feel for how this works:

(1) Where a depends only on x and not on ξ:
(2) When a depends only on ξ and not on x:
(3) Where a has *polynomial dependence* on ξ

[4]Another view of this is that the algebra of differential operators for C^∞ is obtained by a noncommutative version of "tensoring" the algebra of differential operators for the polynomial ring, with $C^\infty(\mathbb{R})$.

The Weyl transform is defined as follows:

Note that both the ordinary and Weyl transform behave in the same way for things that are polynomial in ξ, and this is the prescribed way for any theory of pseudo-differential operators. The difference lies in the *way* we choose to exercise noncommutativity. Roughly, the fact that x and ξ commute in the ring of functions but not in the ring of differential operators, allows us to give myriad ways of interpreting the function $a(x, \xi) = x\xi$ as a differential operator: we could think of it as $x\xi$ as ξx or as the average of the two. The ordinary transform takes the first interpretation (differentiate, then multiply) and is thus not "symmetric". It corresponds to doing things in sequence.

14.3. **Quantizing the transforms.**

14.4. **Determining properties of the transforms and quantizations.**

15. The Hilbert transform

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