DUALITY, MANIFOLDS AND SOME ALGEBRAIC TOPOLOGY

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Abstract. This is a short note intended to explore the applications of duality theory to the study of manifolds. I discuss Alexander duality, Lefschetz duality and Poincare duality, along with applications to the study of compact connected orientable manifolds.

1. Manifolds and points

1.1. **The core question.** One of the questions we shall be interested in is:

> Given two manifolds M and N, what are the ways in which N embeds as a submanifold of M? In other words, what are the submanifolds of M homeomorphic to N?

Roughly speaking, we want to know how N "sits inside" M merely from the data of what M and N look like abstractly. First, we need to define what it means for "ways in which N embeds".

Definition (Equivalently embedded subsets)**.** Give a topological space X and subspaces Y_1 and Y_2, we say that Y_1 and Y_2 are **i equivalently embedded subsets**(defined)f there is a homeomorphism of X under which Y_1 maps homeomorphically to Y_2, or equivalently, there is a homeomorphism of the pair (X, Y_1) and (X, Y_2).

The question we want to ask, more precisely is: what are the various ways of embedding one manifold inside another, upto equivalence?

The answer in general could be *lots*. Moreover, non-equivalent embeddings may also look very similar to the algebraic topologist. Algebraic topology tries to partially solve this problem by looking at an "invariant" of embeddings:

> Given a topological space X and a subspace Y, describe all possibilities for the induced maps of homology:
>
> $$H_i(X \setminus Y) \to H_i(X)$$
>
> Given only the datum of what Y looks like as an abstract topological space.

1.2. **Points in manifolds.** There are some special cases where we can prove that just knowing the abstract homeomorphism type of the manifold and the submanifold, determines a unique way of embedding the submanifold. More precisely, there are special cases of manifolds M and N such that any two embeddings of N in M ar eequivalent. Before beginning this, let's deifne a manifold:

Definition (Manifold)**.** A **manifold**(defined) of dimension n is a topological space such that:

- Every point in the manifold is contained in an open set homeomorphic to Euclidean space.
- The space is Hausdorff
- The space has a countable basis of open sets, viz., it is second-countable.

A topological space which satisfies only the first condition is called a **locally Euclidean space**(defined).

Now the big theorem:

Theorem 1 (Connected manifolds are homogeneous)**.** Let M be a connected manifold. Then any two embeddings of a point in M are equivalent. In other words, given any two points $p, q \in M$, there is a homeomorphism of M which sends p to q.

Proof. Consider the group of self-homeomorphisms of M. This group acts on M, and M is partitioned into orbits. We will prove that each orbit is open. Since M is connected, this will force the action to be transitive.

To prove that each orbit is open, it suffices toshow that every point is contained in an open set such that all points in the open set are in its orbit.

Let $p \in M$ be a point. Then there exists a neighbourhood U of p which is homeomorphic to $\mathbb{R}^n$ via $f : \mathbb{R}^n \to U$. Let V be the image of the interior of the unit disc under f, and D be the image of the unit disc under f. Now, we claim that any point $q \in V$ is in the same orbit as p. To see this:

- Define a map $g_1 : D \to D$ which sends p to q, and is identity on $D \setminus V$ (this can be done; it's a problem in $\mathbb{R}^n$).
- Define a map $g_2 : M \setminus V \to M \setminus V$ given by the identity map.

Now note that:

- D is homeomorphic to the unit disc, so it is compact. Moreover, M is *Hausdorff*, so D is closed in M.
- V is open in U, and U is open in M, so V is open in M. Hence $M \setminus V$ is closed in M.
- g_1 and g_2 agree on $(M \setminus V) \cap D$.

Thus the gluing lemma allows us to get a map $g : M \to M$ which restricts to g_1 on D and g_2 on $M \setminus V$. $g(p) = q$, and we are done. □

The proof crucially relies on the Hausdorffness of M. In general, Hausdorffness is necessary for us to be able to conclude that compact subsets are closed, and this is something we shall keep needing as we proceed.

A locally Euclidean space for which the above proposition fails is the "line with two origins": the real line with two copies of zero.

The proof does not depend on the second-countability assumption. Many of the proofs we shall see do not depend on second-countability. However, second-countability turns out to be necessary to force stronger separation and metrizability properties.

Another theorem, which can be proved along similar lines:

Theorem 2 (Multiple transitivity)**.** Let M be a manifold of dimension greater than 1, and $\{p_1, p_2, \dots, p_r\}$ and $\{q_1, q_2, \dots, q_r\}$ be two sets of points. Then there is a homeomorphism ϕ of M such that $\phi(p_i) = q_i$ for every i.

1.3. **Relative homology for a point.** Let's study the point-deletion inclusion problem, viz., the problem of how the inclusion maps look:

$$H_i(M \setminus p) \to H_i(M)$$

Since (for a connected manifold) any two points in a manifold are equivalently embedded, the nature of the above map should *not* depend on the choice of point. Let's prove our first theorem.

Theorem 3 (Homology of pair with point removed)**.** Let M be a topological space and p a closed point in M, such that p is inside an open neighbourhood U homeomorphic to $\mathbb{R}^n$. Then:

$$H_n(M, M \setminus p) = \mathbb{Z}, \qquad H_i(M, M \setminus p) = 0 \ \forall\ i \neq n$$

Proof. Note that since p is closed, $M \setminus p$ is open. Excision at p yields:

$$H_i(M, M \setminus p) \cong H_i(U, U \setminus p)$$

The result holds for U, since U is homeomorphic to $\mathbb{R}^n$, and hence we are done. $\square$

The proof does not require Hausdorffness. In fact, since locally Euclidean spaces are T_1, the proof works for every point in a locally Euclidean space.

Also, the proof requires a Euclidean neighbourhood only at the particular point. Thus, for manifolds with boundary, the result holds for any point not in the boundary. For CW-complexes, the result holds for points in an attached n-cell, if there is no cell of higher dimension whose boundary contains that point.

The homology of the pair gives a first foot into the problem we're after. We write down the long exact sequence of homology of the pair. We see immediately that for $i \neq n, n-1$, the inclusion:

$$H_i(M \setminus p) \to H_i(M)$$

is an isomorphism. For $i = n, n-1$, we have the picture:

$$0 \to H_n(M \setminus p) \to H_n(M) \to H_n(M, M \setminus p) \to H_{n-1}(M \setminus p) \to H_{n-1}(M) \to 0$$

What happens at n and at $n-1$ depends on the nature of M. This is part of a more general principle: the homology of a pair cares only about local information, whereas the actual inclusion maps care a bit about the global structure as well.

1.4. **Some particular cases.** We assume here the following results:

- If M is any manifold, then $H_i(M) = 0$ for i greater than the dimension of M.
- If M is a compact connected orientable n-manifold, then the map $H_n(M) \to H_n(M, M \setminus p)$ is an isomorphism for all p.
- If M is a compact connected non-orientable n-manifold, $H_n(M) = 0$.
- All the homology groups of a compact connected orientable n-manifold are finitely generated.

Let's see some consequences of this. First, the fact that higher homologies of M being zero, also tells us that higher homologies of $M \setminus p$ are zero.

For a compact connected orientable manifold, we obtain that:

- $H_n(M \setminus p) = 0$
- The map from $M \setminus p$ to M induces an isomorphism on $(n-1)^{th}$ homology

On the other hand, when M is a compact connected non-orientable manifold, we get:

- $H_n(M \setminus p) = 0$
- There is a short exact sequence:

$$0 \to H_n(M, M \setminus p) \to H_{n-1}(M \setminus p) \to H_{n-1}(M) \to 0$$

In sharp contrast to these cases, when $M = \mathbb{R}^n$, then $H_n(M \setminus p) = 0$ and $H_{n-1}(M \setminus p) = \mathbb{Z}$.

2. Alexander duality

2.1. **Statement of Alexander duality.** Alexander duality is a tool for computing the relative homology of the pair $(M, M \setminus K)$ where M is a connected orientable manifold and K is a compact subset of M. We state the result in two parts:

- There is a natural isomorphism:

$$\overline{H}^i(K) \to H_{n-i}(M, M \setminus K)$$

- When K is a submanifold, or when K is a strong deformation retract of an open neighbourhood, we have:

$$H^i(K) \cong \overline{H}^i(K)$$

The precise statement of Alexander duality also gives us the maps, with a very concrete interpretation of those maps. However, we shall not be too interested in the specific maps as of now. Our main objective in looking at Alexander duality is to study the problem of how submanifolds can be embedded inside orientable manifolds.

Alexander duality does for compact submanifolds what out explicit excision argument did for points. In fact, applying Alexander duality for a *point* in a manifold gives exactly the answer we got in the previous subsection.

2.2. **The general problem formulation.** Suppose M is a connected orientable manifold and K is a compact connected submanifold, and we know K upto homeomorphism, but have no clue about how K sits inside M. We want to compute the maps:

$$H_i(M \setminus K) \to H_i(M)$$

Clearly, if K embeds in M in only one possible way (as happens when K is a point) then the maps are uniquely determined. However, there are situations where different possible embeddings of K give rise to the same maps on homology.

In the coming sections, we shall see what can be said under special cases where:

- M is a compact connected orientable manifold
- K is a sphere
- M is a sphere
- M is highly connected, and K has small codimension

2.3. **When the manifold is compact connected orientable.**

Claim. Suppose M is a compact connected orientable manifold and K is a compact connected submanifold of M. Then the map:

$$H_n(M) \to H_n(M, M \setminus K)$$

is an isomorphism.

Proof. Let $p \in K$. Then consider:

$$H_n(M) \to H_n(M, M \setminus K) \to H_n(M, M \setminus p)$$

Note that $H_n(M, M \setminus K)$ is, by Alexander duality, the same as $H^0(K)$, which is $\mathbb{Z}$.

Thus, all three groups are $\mathbb{Z}$, and the composite map is an isomorphism, so both maps are isomorphisms. $\square$

The consequence is that $H_n(M \setminus K) = 0$, and we can start our homology chase from $n-1$ instead of n.

2.4. **When the submanifold is a sphere.** Suppose M is a connected orientable n-manifold and K is homeomorphic to S^m. Then Alexander duality tells us that:

$$H_n(M, M \setminus K) = H_{n-m}(M, M \setminus K) = \mathbb{Z}$$

and all other relative homologies are 0. Thus the inclusion of $M \setminus K$ in M induces isomorphisms on all homologies except possibly $n, n-1$, and $n-m, n-m-1$.

If we are also given that M is compact connected orientable, then $H_n(M \setminus K) = 0$ and we have isomorphisms everywhere else except possibly at $n-m$ and $n-m-1$.

2.5. **When the manifold is a sphere.** If M itself is a sphere, then all the homologies of M vanish except the 0^{th} and n^{th} homology. Also, M is compact connected orientable, so what we get is:

- $H_n(M \setminus K) = 0$
- $H_i(M, M \setminus K) \cong \tilde{H}_{i-1}(M \setminus K)$ for $i \leq n-1$.

2.6. **When both the manifold and submanifold are spheres.** Combining all the results above, we see that when $M = S^n$ and K is homeomorphic to S^m, then $\tilde{H}_i(M \setminus K)$ is $\mathbb{Z}$ for $i = n-m-1$ and 0 elsewhere.

Thus, we have the surprising result that the Alexander duality method cannot distinguish between different ways that S^m may sit inside S^n. This might lead to the question: are any two S^ms inside S^n equivalently embedded? The answer is *no*: The Alexander Horned sphere inside S^3 is homeomorphic to S^2, but it is not embedded in the same way as an equatorial S^2.

There are other interesting consequences. Note that nowhere did we use that K is actually homeomorphic to S^m: all we used was that K is compact and has the same homology groups as S^m. Thus, the results hold for any compact m-manifold with the same homology groups as S^m. Such manifolds are called *homology spheres*, and there exists a homology sphere of dimension 3 which are not homotopy-equivalent to S^3.

2.7. **The circle inside the torus.** Let's now look at a situation where we have multiple ways of embedding a submanifold inside a manifold. Suppose M is the torus, and K is homeomorphic to a circle. There are two obviously distinct ways i nwhich K ca nbe embedded inside M:

- K can be embedded as a circle inside a Euclidean open subset. In this case, $M \setminus K$ has two connected components, one of which is contractible, and the other is homotopy-equivalent to a wedge of two circles.
- K can be embedded as one of the S^1s of $S^1 \times S^1$. The complement of K is now a single connected component, homotopy-equivalent to a circle.

Let's see if Alexander duality gives us both these possibilities, and if it gives any others. Since M is compact connected orientable, the homology chase can start from 1, so we get the following long exact sequence:

$$0 \to H_1(M \setminus K) \to H_1(M) \to H_1(M, M \setminus K) \to \tilde{H}_0(M \setminus K) \to 0$$

Putting known values, we get:

$$0 \to H_1(M \setminus K) \to \mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z} \to \tilde{H}_0(M \setminus K) \to 0$$

Some homological algebra yields two possibilities:

- $H_1(M \setminus K) = \mathbb{Z} \oplus \mathbb{Z}$, with the map from it being an isomorphism, and $\tilde{H}_0(M \setminus K) = \mathbb{Z}$.
- $H_1(M \setminus K) = \mathbb{Z}$, with the map from it being a direct factor embedding, and $\tilde{H}_0(M \setminus K) = Z$.

These are the two possibilities we have already sketched "visually".

2.8. **Connectivity and embeddings.** Here is an interesting result:

Theorem 4 (Compact connected submanifolds are separating)**.** Suppose M is a path-connected simply connected n-manifold, and K is a compact simply connected submanifold of codimension 1. Then K is a separating submanifold: $M \setminus K$ has exactly two connected components. Moreover, K must be orientable.

Proof. We would like to use Alexander duality, but the problem is that we do not know whether the highest cohomology of K is $\mathbb{Z}$. The trick is to use Alexander duality with $\mathbb{Z}/2\mathbb{Z}$-coefficients, because every manifold is $\mathbb{Z}/2\mathbb{Z}$-orientable.

Since K is $\mathbb{Z}/2\mathbb{Z}$-orientable, its top homology is $\mathbb{Z}/2\mathbb{Z}$. Also, since M is $\mathbb{Z}/2\mathbb{Z}$-orientable, Alexander duality applies to M, and we get:

$$\mathbb{Z}/2\mathbb{Z} \cong H^{n-1}(K; \mathbb{Z}/2\mathbb{Z}) \cong H_1(M, M \setminus K; \mathbb{Z}/2\mathbb{Z})$$

We now use a similar argument with the pair $(M, M \setminus K)$. First note that since M is simply connected, $H_1(M) = 0$. For first homology, $H_1(M; R) = H_1(M) \otimes R$, so we get $H_1(M; \mathbb{Z}/2\mathbb{Z}) = 0$. Also $\tilde{H}_0(M; \mathbb{Z}/2\mathbb{Z}) = 0$ since M is connected. Thus the long exact sequence of homology yields an isomorphism:

$$H_1(M, M \setminus K; \mathbb{Z}/2\mathbb{Z}) \to \tilde{H}_0(M \setminus K; \mathbb{Z}/2\mathbb{Z})$$

Combining with the previous observation, we see that the reduced homology of $M \setminus K$ is $\mathbb{Z}/2\mathbb{Z}$.

Now the zeroth reduced homology is free on 1 less than the number of connected components; hence, $M \setminus K$ has exactly two connected components. But now we can go back and put $\mathbb{Z}$-coefficients, and we'll have:

$$\tilde{H}_0(M \setminus K; \mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}$$

Proceeding backwards, we get that the top cohomology of K is $\mathbb{Z}$; hence K is orientable. Note that to proceed backwards, we use the following facts:

- $H_1(M) = 0$ because M is simply connected
- Alexander duality applies, because M is simply connected, and hence orientable

□

This proves, among other things, results like the Jordan separation theorem (set $M = \mathbb{R}^n$, $K = S^{n-1}$). Note that the assumption of M simply connected was crucial; in the previous section, we saw that when M is not simply connected, then we can embed curves of codimension 1 that do not separate M.

3. Homology, cohomology and connected sums

3.1. **Facts.** This is a list of well-known "facts":

(1) If a map from a topological space X to a topological space Y induces isomorphisms on all homologies for $1 \leq i \leq r$, then it also induces isomorphisms on all cohomologies for $1 \leq i \leq r$, and moreover, it preserves the cup product structure for $i, j, i+j \in \{1, 2, \dots, r\}$. The first part follows from naturality of the universal coefficient theorem; the second part follows from the fact that the cup product commutes with homomorphisms, which in turn follows from naturality of the Alexander-Whitney map.
(2) The cohomology ring of a wedge sum is the direct sum of the cohomology rings (preserving gradation) modulo an identification of the zeroth cohomology group. One can see this, for instance, from the fact that the map from the disjoint union to the wedge sum induces isomorphisms on all positive homologies.
(3) The cohomology ring of a connected sum of compact connected orientable manifolds is the direct sum of the cohomology rings (preserving gradation) modulo an identification of the zeroth cohomology group, and an identification of the n^{th} cohomology group. We shall discuss this in a little more detail.

3.2. **Homology of a connected sum.** For now, let M_1 and M_2 be connected manifolds.

There are many approaches to computing the homology of a connected sum; perhaps the most elementary is a Mayer-Vietoris. The upshot of the initial Mayer-Vietoris computation is that for $i \neq n, n-1$, we have isomorphisms:

$$H_i(M_1 \sharp M_2) \cong H_i(M_1) \oplus H_i(M_2)$$

To prove this, we use two basic facts:

- The inclusion of $M_i \setminus p$ in M_i induces isomorphism on all homologies except possibly $n, n-1$.
- The only place where the intersection of the open sets (which is effectively S^{n-1}) has nonzero reduced homology is at $n-1$. Hence it can affect only $n, n-1$.

Again, the behaviour of the connected sum at $n, n-1$ depends on the nature of the manifolds. When M_i is compact connected orientable, then the inclusion of the glued S^{n-1} into M_i is nullhomotopic (since S^{n-1} is assumed to live inside a Euclidean neighbourhood, and the inclusion factors through that). Hence, we get that the inclusion of S^{n-1} in the manifold minus a point is also nullhomotopic, and using this, we see that if both manifolds are compact connected orientable:

- The connected sum is also compact connected orientable
- The $(n-1)^{th}$ homology of the connected sum is the sum of the $(n-1)^{th}$ homology of the pieces.

It turns out that the $(n-1)^{th}$ homology of the connected sum remains the sum of the homologies of the pieces, even if *one* of the manifolds is non-orientable; however, the result fails if both are non-orientable.

3.3. **Maps from the connected sum to the pieces.** Suppose M and N are manifolds. Then there is a map from $M \sharp N$ to M, given by collapsing the whole of N to a point. (The manifold we get is not M itself, but is homeomorphic to M via shrinking inside a Euclidean neighbourhood). There is a similar map to N.

These maps have some very nice properties, as we shall see. First, a little definition:

Definition (Degree of a map)**.** Let M and N be compact connected orientable manifolds, and let $[M]$ and $[N]$ be fundamental classes for M and N respectively (viz., generators of the top homologies). Then if $f : M \to N$ is a continuous map, the degree of f is defined as the unique integer d such that $[M]$ maps to $d[N]$.

Given compact connected orientable manifolds, one question is whether there exists a map of degree ± 1 between them (a map of degree 1 can be converted to a map of degree -1 by reversing the orientation on one side). It turns out that the existence of degree one maps poses fairly strong restrictions; in some sense M must have "more complexity" than N for there to exist a degree 1 map from M to N.

Theorem 5 (Degree one maps from connected sum)**.** The map $M\sharp N \to M$ obtained by pinching N to a point, is a degree one map.

Proof. Pick a point p anywhere in M outside the part which got pinched. Then the inverse image of p in $M\sharp N$ is also one point, and the map at the inverse image is a local homeomorphism. Call the inverse image q. Consider the following commutative diagram:

$$\begin{array}{ccc} M\sharp N & \to & (M\sharp N, M\sharp N \setminus q) \\ \downarrow & & \downarrow \\ M & \to & (M, M \setminus p) \end{array}$$

The map on the right is an orientation-preserving local homeomorphism so it induces isomorphism on top homology. The horizontal arrows induce isomorphism on top homology, so the left vertical arrow should have degree 1 (if we gave our orientations correctly). ☐

The proof idea generalizes to computing local degrees: given any point p in the target space, look at $f^{-1}(p)$. If $f^{-1}(p)$ is a discrete set, we can use an excision argument to see that the total degree is the sum of the local degrees at each point. If the map is a local homeomorphism at each point in $f^{-1}(p)$, then the total degree is the size of the inverse image. This is what happens, for instance, in the case of covering maps. Incidentally, this also shows that any map of nonzero degree must be surjective.

A particular case of a degree one map from the connected sum is the case where we take M to be S^n. Then we get a degree one map from $S^n\sharp N$ to S^n, and since N is the same as $S^n\sharp N$, we get a degree one map from any compact connected orientable manifold to S^n. In fact, we shall see soon that S^n is a terminal object in the category of compact connected orientable manifolds with degree one maps.

To get a better understanding of compact connected orientable manifolds, we need to look a bit at Poincare duality.

4. Poincare duality

4.1. **Statement of Poincare duality.**

Poincare duality is a special case of Alexander duality when the manifold M itself is compact, and hence we can set $K = M$. Thus, Poincare duality tells us that:

$$H^i(M) \cong H_{n-i}(M)$$

Here are some corollaries of Poincare duality:

1. The top homology and the top cohomology of M are $\mathbb{Z}$. This can be proved by taking $i = 0$ and $i = n$ respectively, and using the fact that M is connected.
2. The second highest homology group of M is free Abelian. This is because it is isomorphic to the first cohomology group, which is free Abelian, since it is obtained as the group of homomorphisms from an Abelian group to $\mathbb{Z}$.
3. The Betti numbers of M are symmetric. In other words, the rank of the k^{th} homology group equals the rank of the $(n-k)^{th}$ homology group.

More is true. To see this, we need to look at the Poincare duality map carefully.

The Poincare duality map is the map from $H^i(M)$ to $H_{n-i}(M)$. The map sends $a \in H^i(M)$ to $a \cap [M]$, where $[M]$ is a fundamental class of M. This map is unique up to a choice of fundamental class.

To understand the Poincare duality map, we note some facts about cap products.

4.2. **Cap products.**

The cap product is a structure by which homology becomes a module over cohomology; it is a collection of maps:

$$H^i(X) \otimes H_j(X) \to H_{j-i}(X)$$

The cap product satisfies a "naturality" condition, which is interesting because one of the functors is covariant and the other is contravariant. Suppose $f : X \to Y$ is a continuous map.

Then $H^*(X)$ admits $H_*(X)$ as a module. Via the induced map from $H^*(Y)$ to $H^*(X)$, we see that $H_*(X)$ becomes a $H^*(Y)$-module. $H_*(Y)$ is already a $H^*(Y)$-module, and the naturality condition is that the natural map from $H_*(X)$ to $H_*(Y)$ is a map of $H^*(Y)$-modules.

We now prove an easy result:

Claim. Consider a topological space obtained as $A \vee B$. Then the cap product of any nonzero cohomology class coming from A, with any nonzero homology class coming from B, is 0.

Proof. Let $X = B$, $Y = A \vee B$ and $f : X \to Y$ be the inclusion. Suppose h is a cohomology class coming from A. From observations we made about the cohomology ring of a wedge sum, it is clear that h gets mapped to 0 under f^*.

Now consider the action of $H^*(Y)$ on $H_*(X)$. h must act trivially, because its image $f^*(h)$ in $H^*(X)$ is 0. But now f_* is a $H^*(Y)$-module map, and hence $h \in H^*(Y)$ acts trivially on any element of $H_*(Y)$ coming from $H_*(X)$. This proves the claim. □

This immediately rules out "wedge-type" structures from being homotopy-equivalent to compact connected orientable manifolds. For instance, $S^2 \vee S^6 \vee S^8$ cannot have the homotopy type of a compact connected orientable manifold, because the cap product of the second cohomology with the top cohomology is trivial.

4.3. **More from cap products.** Cap products can actually tell us a fair deal about compact connected orientable manifolds. Let us return to the theme of studying the degrees of maps between compact connected orientable manifolds. So far, we have noted the following:

(1) Given two compact connected orientable manifolds, we can compute the degree of a map between them via a "local degree" computation: pick a point, look at the set of its inverse images, and compute the degree of the map in neighbourhoods at each of the inverse images. This works if the set of inverse images is discrete.
(2) In particular, a covering map of finite degree d has degree d as a map of compact connected orientable manifolds.
(3) The map from a connected sum to either piece obtained by pinching the other piece, has degree 1.
(4) Given any manifold, there is a map from the manifold to the sphere, of degree 1.

We are now interested in developing tools whereby one can conclude that there does *not* exist a degree one map between compact connected orientable manifolds. Cap products give us such a tool. Suppose M and N are compact connected orientable manifolds with chosen fundamental classes $[M]$ and $[N]$, and suppose $f : M \to N$ is a continuous map which sends $[M]$ to $[N]$. Then we get the following maps:

$$H^i(N) \to H^i(M) \to H_{n-i}(M) \to H_{n-i}(N)$$

Here, the middle map involves capping with the fundamental class of M.

Moreover, what we said earlier about the "naturality" of cap products *and* the fact that the fundamental class $[M]$ gets mapped to the fundamental class of N, tells us that the composite is capping with $[N]$. But since the composite is an isomorphism, we see that the left-most map must be injective and the right-most map must be surjective.

Thus, the cohomology of N is "smaller" than the cohomology of M and the homology of M is "bigger" than the homology of N (both statements are consistent with each other). We can view degree one maps as a kind of "going-down" in complexity, as far as the amount of homology is concerned. Some more facts which can be deduced along similar lines:

- If $f : M \to N$ is a degree one map of compact connected orientable manifolds, then the induced map on the fundamental groups is also surjective.
- Suppose there exists a degree one map from M to N and a degree one map from N to M. Then the composite of these maps induces a surjective map from $H_*(M)$ to $H_*(M)$. Since all the homology groups are finitely generated, the surjective map must be an isomorphism, and hence the composite induces isomorphisms on the homology of M. It also induces an isomorphism on

the fundamental group of M, and hence turns out to be a homotopy equivalence from M to itself (by Whitehead's theorem). Similar reasoning shows that the composite at the N-end is also a homotopy-equivalence, so M and N are homotopy-equivalent.

- Thus, we can define a partial order on homotopy classes of compact connected orientable manifolds, where $M \geq N$ iff there exists a degree one map from M to N.
- If $M \geq N$, then each Betti number of M is greater than or equal to the corresponding Betti number of N.

4.4. **The partial order for surfaces.** For compact connected orientable surfaces, we get a total ordering. A classification theorem tells us that the only compact connected orientable surfaces are the surfaces of genus g, where a surface of genus g is a connected sum of g tori (the surface of genus 0 is the sphere).

Let M_g denote the surface of genus g. Then for $g \geq h$, there exists a degree 1 map from M_g to M_h, because $M_g = M_h \sharp M_{g-h}$. Moreover, there cannot exist a degree one map from M_h to M_g if $g > h$, because the first Betti number of M_g is $2g$, and the first Betti number of M_h is $2h$.

Thus, the partial order we described in the previous subsection yields a total ordering for surfaces. In fact, more is true. The argument of the previous subsection generalizes to any ring of coefficients; hence, we conclude that there does not even exist a map from M_h to M_g which induces an isomorphism on the top *rational* homology if $g > h$. Equivalently, there does not exist a map of nonzero degree from M_h to M_g if $g > h$.

5. Constructing and studying compact connected orientable manifolds

We now enter the murky waters of compact connected orientable manifolds in higher dimensions than 2. While anything resembling a classification is far beyond the scope here, we would at least like to get a reasonable number of manifolds and study their properties. Let us recall the various techniques for constructing compact connected orientable manifolds:

(1) Connected sums: This is a technique for *adding up* the homological complexity (as in, adding up the number of holes in each dimension). To take a connected sum, we need two manifolds of the same dimension, and we would like both of them to be orientable.
(2) Product: A product of compact connected orientable manifolds is compact connected orientable. The dimensions add up in a product, so this can be a tool for constructing higher-dimensional compact connected orientable manifolds from lower-dimensional ones.
(3) Spheres: The starting point of all homology theory is the spheres. These are the "simplest" compact connected orientable manifolds.
(4) Gluing cells: A more complicated construction than taking products is to glue cells in non-conventional ways. This is how, for instance, complex projective spaces are constructed. That takes us into the realm of CW-complexes.

These constructions can be used to construct manifolds that are non-compact and non-orientable as well (provided we start out with some that aren't) but we'll restrict attention to the relatively smaller class of compact connected orientable manifolds.

Our goal will be to relate the geometric, algebraic and topological effect of each of the operations, and hopefully gain some intuition into the way each operation works.

5.1. **Products.** There are many ways to understand how products work. Let's first do it using cell structures (as in CW-complex structures).

Suppose X and Y are (sufficiently nice) topological spaces, each of which is given a certain cell structure. Then $X \times Y$ also gets a natural cell structure. What happens is that for each i-cell of X and each j-cell of Y, we get a corresponding $(i+j)$-cell of $X \times Y$. A convenient way to compute the number of n-cells in the product is by considering generating functions of the number of cells in X and the number of cells in Y; specifically, define:

$$p_c(X) = \sum_i \alpha_i x^i$$

where α_i is the number of i-dimensional cells of X. Then we have:

$$p_c(X \times Y) = p_c(X) p_c(Y)$$

The following fact is true (we will note prove it, because it will take us too far afield):

> If all the homologies of a CW-complex are free Abelian, then we can construct a homotopy-equivalent CW-complex, where all the boundary maps are zero. In fact, we can give the CW-complex a cell structure where the boundary of every k-cell lies inside the $(k-2)$-skeleton (except at $k=1$, of course). With this new CW-complex structure, the cellular chain groups are precisely the homology groups.

Thus, a good model for spaces with free homology are spaces where all the boundary maps go into the $(k-2)$-skeleton. For such a cell structure, the number of cells in dimension i is precisely equal to the rank of the i^{th} homology group.

Definition (Poincare polynomial)**.** The Poincare polynomial of a topological space with finitely generated homology, is the generating function for the Betti numbers. In other words, it is a polynomial where the coefficient of x^j is the Betti number b_j.

For spaces with free homology, the Poincare polynomials multiply. For instance, the Poincare polynomial of the sphere S^n is $1+x^n$, and the Poincare polynomial for a product of spheres $S^{m_1} \times S^{m_2} \times S^{m_r}$ is:

$$(1+x^{m_1})(1+x^{m_2})\dots(1+x^{m_r})$$

In particular, the Poincare polynomial for $S^1 \times S^1 \times \dots S^1$ is:

$$(1+x)^n$$

Hence the k^{th} Betti number is $\binom{n}{k}$.

There is a general theory for what happens if the spaces involved do not have free homology, but in that case, we need to actually look at the homology groups, and cannot simply work with Betti numbers. The formula that we get is termed the "Kunneth formula", which involves taking tensor products of the homology groups and adding. The "tensor product" operation is analogous to multiplication of monomials and the "adding up" is analogous to adding up monomials to get a polynomial.

Note that the Euler characteristic is the value taken by the Poincare polynomial at -1. This shows us immediately that for a product of circles, the Euler characteristic is always zero. On the other hand, for a product of arbitrary spheres, the Euler characteristic is zero iff at least one of the spheres is odd-dimensional.

As we proceed, we shall see an interesting even-versus-odd bifurcation in properties.

Let's also describe the cohomology ring of a product. The cohomology ring of a product is a graded tensor product of the cohomology rings of each of the pieces. In the particular case where the pieces are $S^{m_1}, S^{m_2}, \dots, S^{m_r}$, the cohomology ring looks like:

$$\mathbb{Z}\langle e_1, e_2, \dots, e_r\rangle / \langle\langle e_i e_j - (-1)^{m_i m_j} e_j e_i, e_i^2 \rangle\rangle$$

We're essentially taking the associative algebra generated by the cohomology rings, modulo the commutation relations that need to be satisfied because of degree constraints. Each e_i lives in degree m_i, and is the generator for the cohomology that "comes from" the projection onto the sphere S^{m_i}.

Some further points:

- For a product of spheres, whenever m_i is even, e_i is in the center. Thus, if the product is only of even-dimensional spheres, then we get a quotient of the polynomial ring in r variables, by the ideal generated by the squares of the variables:

$$\mathbb{Z}\langle e_1, e_2, \dots, e_r\rangle / \langle\langle e_i^2 \rangle\rangle$$

- If all the m_is are odd, then we get an exterior algebra on the e_is – an algebra generated by anticommuting elements, namely:

$$\mathbb{Z}\langle e_1, e_2, \dots, e_r\rangle / \langle\langle e_i^2, e_i e_j + e_j e_i \rangle\rangle$$

  The e_i here lives in degree m_i.

5.2. **Connected sum.** When we take a connected sum, we add up the homologies in each of the dimensions, except in the *top* dimension, where the fundamental classes coming from each piece get identified.

Intuitively, we are "adding up" the number of holes in all dimensions except the top dimension. The reason why the number of holes in the top dimension doesn't get added up is because we are gluing along a S^{n-1}.

Thus a connected sum is a good way to "add up" the complexity of two manifolds.

One point deserves mention here. When we take a connected sum of two manifolds, there are two possible ways of gluing together the sphere of one manifold with the sphere of the other. These two ways could give manifolds which are not even homotopy-equivalent.

This non-equivalence is reflected in the structure of the cohomology ring. The choice of how to glue together the S^{n-1}s is related to the choice of how to identify the fundamental class coming from one piece, with the fundamental class coming from the other piece. For certain kinds of manifolds, both choices give the same ring, whereas for certain choices of manifolds, we could get two non-isomorphic ring.

Let's first see what's happening for products of spheres, in which case connected sum *is* unique – this is essentially because spheres possess orientation-reversing homeomorphisms. Consider the connected sum of $S^2 \times S^2$ and $S^1 \times S^3$.

The cohomology ring of $S^2 \times S^2$ looks like:

$$\mathbb{Z}[e_1, e_2]/(e_1^2, e_2^2)$$

Here both e_1 and e_2 live in degree 2.

The cohomology ring of $S^3 \times S^1$ looks like:

$$\mathbb{Z}\langle e_3, e_4\rangle/\langle\langle e_3^2, e_4^2, e_3e_4 + e_4e_3\rangle\rangle$$

When we take a connected sum, we need to ensure that a product of anything from one piece with anything from the other piece is zero; we also need to identity the top cohomology classes. The end result is this:

$$\mathbb{Z}\langle e_1, e_2, e_3, e_4\rangle/I$$

where I is the two-sided ideal generated by:

$$e_1^2, e_2^2, e_3^2, e_4^2, e_1e_2 - e_2e_1, e_3e_4 + e_4e_3, e_1e_3, e_2e_3, e_1e_4, e_2e_4, e_1e_2 - e_3e_4$$

The last relation identifies the top cohomology classes in both pieces.

Taking products is like multiplying complexity; taking connected sums is like adding complexity within the same dimension. We now consider the third construction, which is all about gluing things in a way that cannot be "decomposed".

5.3. **The projective spaces.** Let $\mathbf{k}$ be one of $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$. Then one can talk of $\mathbf{k}^{n+1}$ as the standard $n+1$-dimensional vector space over $\mathbf{k}$ (for the case of $\mathbb{H}$, one must talk of "left vector space"). Inside $\mathbf{k}^{n+1}$, one can look at the "sphere" $S^n(\mathbf{k})$, which is the set of elements $(z_1, z_2, \dots, z_{n+1})$ such that:

$$\sum_{i=1}^{n} |z_i|^2 = 1$$

The following are true:

- By treating $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{H}$ as $\mathbb{R}^1$, $\mathbb{R}^2$ and $\mathbb{R}^4$, we can identify $S^n(\mathbb{R})$ with S^n, $S^n(\mathbb{C})$ with S^{2n+1}, and $S^n(\mathbb{H})$ with S^{4n+3}.
- $S^0(\mathbf{k})$ is the group of units in $\mathbf{k}$ which have norm 1. $S^0(\mathbb{R})$ is the group $S^0 = \mathbb{Z}/2\mathbb{Z}$, $S^0(\mathbb{C})$ is the group S^1, and $S^0(\mathbb{H})$ is the group S^3 of unit quaternions.
- There is a natural action of $S^0(\mathbf{k})$ on $S^n(\mathbf{k})$ by coordinate-wise multiplication, and the quotient of $S^n(\mathbf{k})$ by this action is the so-called "projective space" $\mathbf{k}\mathbb{P}^n$. In other words, we get fiber bundles:

$$\begin{aligned} S^0 &\to S^n &&\to \mathbb{RP}^n \\ S^1 &\to S^{2n+1} &&\to \mathbb{CP}^n \\ S^3 &\to S^{4n+3} &&\to \mathbb{HP}^n \end{aligned}$$

- The quotient $\mathbf{k}\mathbb{P}^1$ is the one-point compactification of $\mathbf{k}$, which is a sphere: $\mathbb{RP}^1 = S^1$, $\mathbb{CP}^1 = S^2$ and $\mathbb{HP}^1 = S^4$. We thus get three fiber bundles of the form:

$$\begin{aligned} S^0 &\to S^1 &&\to S^1 \\ S^1 &\to S^3 &&\to S^2 \\ S^3 &\to S^7 &&\to S^4 \end{aligned}$$

Equipped with this knowledge, let us try to realize projective spaces over $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{H}$ in terms of CW-complexes. Explicitly:

- The projective space $\mathbf{k}\mathbb{P}^n$ is constructed from $\mathbf{k}\mathbb{P}^{n-1}$ via the attaching map $S^n(\mathbf{k}) \to \mathbf{k}\mathbb{P}^{n-1}$.
- For real projective space, we are attaching a n-cell to the $(n-1)$-skeleton. For complex projective space, we are attaching a $2n+2$-cell to the $2n$-skeleton, and for Hamiltonian projective space, we are attaching a $4n+4$-cell to the $4n$-skeleton.
- Thus, for complex projective space, the cellular chain complex has homology group $\mathbb{Z}$ in even dimensions upto $2n$ and 0 in odd dimensions. For Hamiltonian projective space, the cellular chain complex has homology group $\mathbb{Z}$ in dimensions $4, 8, \ldots, 4n$ and 0 elsewhere. Real projective spaces require a bit more of computation for homology which we won't do.
- All projective spaces are compact. This is because they are quotients of spheres, which are already compact.
- In fact, complex and Hamiltonian projective spaces are simply connected, because they have no 1-cells. Thus, they are orientable. This gives us some genuinely new examples of compact connected orientable manifolds.
- Real projective spaces are orientable iff the dimension is odd. One way of seeing this is that the real projective space is obtained by quotienting the sphere by the antipodal identification; and the antipodal identification is orientation-preserving iff the dimension is odd.
- The cellular structure of projective spaces is pretty rigid: each new cell that we are adding has its boundary mapping surjectively to the previous skeleton. We shall see that this is closely related to the fact that in the cohomology ring, the lowest cohomology generates all the higher cohomologies.

So far, we have discussed the operation of taking products and connected sums. Products had more complexity, but the complexity was multiplicatively decomposed; connected sums had complexity, but the complexity was additively decomposed. However, constructions like those of the complex projective space are *indecomposable* because each layer of complexity is built into the previous one.

Let's look at the cohomology rings. The cohomology ring of $\mathbb{CP}^n$ is $\mathbb{Z}[x]/(x^{n+1})$ where x is a generator of the second cohomology. In other words, generators of all the higher cohomology groups, are obtained by taking cup powers of $\mathbb{CP}^n$. Some remarks:

- Any map from $\mathbb{CP}^n$ to itself, which is multiplication by d on the second cohomology, is multiplication by d^k on the $2k^{th}$ cohomology. Hence its Lefschetz number is $1 + d + d^2 + \ldots + d^n$ and its degree (as a map of compact connected orientable manifolds) is d^k.
- If a map from $\mathbb{CP}^n$ to itself takes the 2-skeleton to itself, then the effect of the map on the cohomology of the 2-skeleton, determines the effect of the map on the whole cohomology ring. In other words, if it is multiplication by d on the cohomology ring of the 2-skeleton, then it is multiplication by d^k on the $2k^{th}$ cohomology.
- The first observation tells us that $\mathbb{CP}^n$ can possess an orientation-reversing homeomorphism only if n is odd. Also, combined with the Lefschetz fixed-point theorem, it tells us that $\mathbb{CP}^n$ has the fixed-point property if n is even. That the converses of these hold requires explicit constructions of maps.

These two facts may seem surprising, but what's going on is that what happens on the "outermost" part determines more or less what happens everywhere, as far as the cohomology ring is concerned.

We see that looking at the cohomology ring, and the cup product structure, already gives us significantly more constraints than merely looking at the effect on the homology groups individually.

6. Manifolds and maps

In this section we study the problem of manifolds, self-maps of manifolds, and maps between manifolds. Some of the questions we shall be concerned with:

- Given a topological space, what do its homology groups and cohomology ring look like?
- Given a topological space, what are the possible graded ring endomorphisms of its cohomology ring? Which of these can be realized by a continuous map, and to what extent does the effect of the map on the cohomology ring, determine its homotopy type?
- Similar questions, but for maps between different topological spaces.

The typical approach we will use is to restrict our attention to what the map does in the dimensions which generate the cohomology ring. For instance, if the cohomology ring is, as a ring, generated by the first cohomology group, then we shall restrict attention to possible endomorphisms of the first cohomology group, and ask which such endomorphisms extend to ring endomorphisms.

6.1. **A review of compact connected orientable surfaces.** Essentially there was only one kind of compact connected orientable surface: the surface of genus g. The cohomology ring of this surface in general looks like:

$$\mathbb{Z}\langle x_1, y_1, \ldots, x_g, y_g \rangle / \langle\langle x_i y_i + y_i x_i, x_i^2, y_i^2, x_i x_j, x_i y_j, y_i y_j, x_i y_i - x_j y_j \rangle\rangle$$

All the x_i and y_i live in degree 1.

Any self-map of a topological space induces an endomorphism of the cohomology ring as a graded ring. Some thought reveals that the endomorphisms of this graded ring correspond to maps which preserve an alternating nondegenerate bilinear form on $\mathbb{Z}^{2n}$. Here the (x_i, y_i) are paired together under the bilinear form.

6.2. **The product of circles.** For the product of many copies of the circle, the corresponding cohomology algebra is the exterior algebra in the generators for each circle, and once again, any map is determined by its effect on first cohomology. In this case, it turns out that *any* $\mathbb{Z}$-module map on the first cohomology, can be achieved via a continuous map from the product of circles to itself.

6.3. **Product of even-dimensional spheres.** For $S^2 \times S^2$, the cohomology ring looks like:

$$\mathbb{Z}[e_1, e_2]/(e_1^2, e_2^2)$$

Something interesting happens here: not every $\mathbb{Z}$-module map of second cohomology extends to a continuous map from $S^2 \times S^2$ to itself. The constraint is that since e_1 and e_2 are nilpotent of order 2, they must get sent to nilpotents of order 2, and the only nilpotents of order 2 are multiples of e_1 and e_2. Thus, the map must send each one to a multiple of one of them.

The crucial difference with the odd-dimensional case is that the generators here *commute*, and thus not every element's square is zero. When they *anti-commute*, we could deduce that the square of any combination of them is zero.

6.4. **Some more compact connected orientable manifolds.** We have seen most of the basic operations that go into constructing compact connected orientable manifolds; it's now time to put some of these operations together and see what are some possibilities for 3-manifolds:

(1) There's S^3 (which has the minimum possible homology, and trivial fundamental group, so it's the "smallest" compact connected orientable manifold). The cohomology ring of S^3 is $\mathbb{Z}[x]/x^2$ where x is in degree 3.

    The only endomorphisms of this ring are those which take x to λx: each of these can be realized by an explicit continuous map; in fact, we can construct the map inductively using the fact that S^n is the suspension of S^{n-1}.

    S^3 is, as a topological group, the same as $SU(2)$. It is also the unit quaternion group, and is also the spin group $Sp(3)$.

(2) There's $S^1 \times S^2$. The cohomology ring is $\mathbb{Z}[x_1, x_2]/(x_1^2, x_2^2)$. As one can see from the cohomology ring, the top homology (and cohomology) are controlled by what happens in the first and second cohomology. Moreover, the first cohomology must go to itself, and the second cohomology must go to itself.

    Any pair of $\mathbb{Z}$-module maps separately on the first and second cohomology can be realized by a continuous map.

(3) There's $S^1 \times S^1 \times S^1$, which is a product of circles. The cohomology ring is an exterior algebra in three variables, and any linear map on the first cohomology can be realized by a continuous map.
(4) There's $\mathbb{RP}^3$, which does *not* have free homology and is not simply connected. $\mathbb{RP}^3$ is the topological group $SO(3)$, and its double cover S^3 is also its universal cover. We shall not concern ourselves with the study of $\mathbb{RP}^3$ for now.

What I've listed above are some examples of *prime* manifolds: manifolds which cannot be expressed as connected sums of two "smaller" manifolds. In the partial ordering between them, S^3 is at the bottom. Slightly higher is $S^2 \times S^1$, and $S^1 \times S^1 \times S^1$ is even higher (note that a distributivity law holds between connected sums and products, so the connected sum of $S^2 \times S^1$ and $S^1 \times S^1 \times S^1$ is just $S^1 \times S^1 \times S^1$).

Incomparable with all of these is $\mathbb{RP}^3$ (this can be seen from criteria of surjectivity on homology for degree one maps).

There are other examples of compact connected orientable manifolds, many of which arise by taking quotients of S^3 by finite subgroups of itself. Some of these ar equotients of $\mathbb{RP}^3$ by finite subgroups of that, viz. quotients of $SO(3)$ by finite subgroups thereof. The classification of finite subgroups of $SO(3)$ gives a reasonable list of compact connected orientable manifolds (all of them are orientable because action of a group on itself by left multiplication is orientation-preserving).[1] The quotient of $SO(3)$ by the action of A_5 is also the quotient of S^3 by the action of the group $\Gamma(2,3,5) = SL(2,5)$.

Theorem 6 (Homology sphere)**.** The quotient of S^3 by the action of $SL(2,5) = \Gamma(2,3,5)$, which is the double cover of $A_5 \subset SO(3)$, is a homology sphere.

Proof. Note that the quotient is compact connected orientable.

The fundamental group of the quotient is $SL(2,5)$. Since $SL(2,5)$ equals its own commutator subgroup, the first homology of the quotient is zero. Hence, its first cohomology is 0. Poincare duality tells us that the second homology is 0, so the quotient is a homology sphere. □

The famous lens spaces are also achieved as finite quotients of S^3 by the action of certain subgroups.

7. A return to embedding problems

Let's return to the original problem that we were considering: given an abstract manifold M (say, connected and orientable) and an abstract compact connected manifold K, what are the possible ways (upto equivalence) in which we can embed K in M?

Here are some constraints we have derived (or which are "obvious"):

- The dimension of K has to be strictly less than that of M for there to exist any embedding. This follows from Alexander duality, or from more direct arguments.
- If K is one-dimensional (viz., a point) then there is only one embedding of K in M (upto equivalence). The induced maps on homology by the inclusion of the complement at $n, n-1$ depend upon the nature of M. Elsewhere, they are isomorphisms.
- If K has codimension one, and M is *simply connected*, then K must be orientable. Further, K must separate M.
- If K is a sphere of dimension m, the only possibilities for homology of the complement are at $n, n-1, n-m, n-m-1$. If M is also compact, connected and orientable, we have homology only at $n-m$ and $n-m-1$.
- If M is a sphere, the $(n-i-1)^{th}$ homology of the complement of K equals the i^{th} cohomology of K, for a suitable range of i.

Let us now concentrate on a very specific class of embedding problems: what are the ways in which a compact connected n-manifold embeds inside $\mathbb{R}^{n+1}$? Note that this is equivalent to studying embeddings inside S^{n+1}, since any n-manifold in S^{n+1} must miss a point.

Since $\mathbb{R}^{n+1}$ (or S^{n+1}) is simply connected, we see that any submanifold of dimension n must be orientable, and must separate the space into two pieces. Thus, manifolds like $\mathbb{RP}^2$ or the Klein bottle, which are non-orientable, cannot be embedded in the real world.

Further, we have that if K is the submanifold:

[1]In fact, all the manifolds have trivial tangent bundle

$$H_{n-i-1}(S^{n+1} \setminus K) \cong H^i(K)$$

Thus, even if there are many inequivalent embeddings, the homology groups of the complement are determined uniquely by the homology groups of the manifold. For instance, if we try to embed compact connected orientable surfaces inside S^3, then the above formula yields:

$$H_1(S^3 \setminus K) \cong H^1(K) = H_1(K)$$

This is the "intuitive" statement that the number of one-dimensional holes in a surface of genus g is the number of one-dimensional holes in its complement.

However, homology with $\mathbb{Z}$-coefficients seems to have limitations: while it seems to yield the homology groups readily, it does not impose any further constraints on which manifolds can be embedded. Recall that the proof that any codimension one submanifold must be separating, and hence orientable, involved a mix of $\mathbb{Z}$ and $\mathbb{Z}/2\mathbb{Z}$-coefficients. Thus, it is sometimes helpful to look at other coefficient rings to get more constraints on which submanifolds can be embedded, and how.

Thus, we are currently unable to answer questions like:

- Can real projective space of odd dimension n be embedded in S^{n+1}(the answer is *no*, for $n > 1$)?
- Can complex projective space be embedded inside a sphere, with codimension 1 (the answer is again *no*)?

8. Special kinds of compact connected orientable manifolds

8.1. **Parallelizable manifolds.**

Definition (Parallelizable manifold)**.** A differential manifold is termed **parallelizable**(defined) if its tangent bundle is trivial; viz, it can be given a global coordinate system.

Parallelizability is a condition that makes sense only for manifolds with a differential structure; however, it implies orientability, which makes sense for a manifold even without giving it a differential structure. All the manifolds that we have constructed do have natural differential structures, so the question of parallelizability makes sense. It turns out that the following are true:

- Any Lie group is parallelizable (we shall see Lie groups a little more in coming sections)
- A product of parallelizable manifolds is parallelizable
- Any quotient of a connected Lie group by the action of a finite subgroup by left multiplication is again parallelizable

Since S^1 and S^3 are Lie groups, we get a wide range of parallelizable manifolds: all products of S^1s and S^3, as well as all quotients of S^3 by finite subgroups, which includes $SO(3)$, the homology 3-sphere, and the lens spaces.

8.2. **Lie groups.**

Definition (Lie group)**.** A **Lie group**(defined) is a manifold with a compatible group structure. It turns out that any Lie group can be equipped with a differential structure such that the group multiplication is smooth for that structure (this is far from obvious).

The remarkable thing about Lie groups, as opposed to arbitrary manifolds, is that they are much more homogeneous, and they have a lot more of symmetry. Here are some applications:

Theorem 7 (Euler characteristic of Lie group)**.** Any compact connected nontrivial Lie group, has zero Euler characteristic.

We assume here the fact that any compact connected nontrivial Lie group admits a simplicial complex structure.

Proof. Let G be the group, and e its identity element. Pick $g \neq e$. Then left multiplication by g is in the same homotopy class as the identity map (a homotopy is given by a path from e to g). Also, left multiplication by g has no fixed points. Hence, the identity map is homotopy-equivalent to a map which has no fixed points, and thus the Lefschetz number of the identity map is zero.

(Here we are applying Lefschetz fixed-point theorem, and using the fact that any compact connected Lie group admits a triangulation)

The Euler characteristic is the same as the Lefschetz number of the identity map; thus we have proved that the Lefschetz number of the identity map is zero. □

Also, another result:

Theorem 8 (Lie groups are parallelizable)**.** Any Lie group is parallelizable.

Here we are using the definition of a Lie group as a differential manifold with a compatible group structure. The proof uses the fact that left multiplication by group elements can be used to translate a basis of the tangent space at the identity, to a basis at all points.

Thus, compact connected Lie groups are examples of compact connected orientable manifolds with Euler characteristic zero. The Euler characteristic does not impose additional constraints for *odd*-dimensional manifolds, because for a compact connected orientable manifold, the Euler characteristic is anyway zero (the Betti numbers match up by Poincare duality). On the other hand, we do get a restriction on the Betti numbers for even-dimensional manifolds. For instance, we see trivially that even-dimensional spheres, and even-dimensional real projective spaces, cannot be Lie groups.

Also, complex and quaternionic (Hamiltonian) projective spaces cannot be Lie groups.

Here is yet another fact about Lie groups:

Theorem 9. A compact connected nontrivial Lie group possesses an orientation-reversing homeomorphism. Further, if there are precisely d d^{th} roots of unity, then the map $x \mapsto x^d$ induces multiplication by d on the fundamental class.

Thus compact connected (nontrivial) Lie groups are parallelizable (hence orientable), have Euler characteristic zero, and possess orientation-reversing homeomorphisms.

8.3. **Compact simply connected manifolds.** Another important class of compact connected orientable manifolds is ithose which are *simply connected*. For instance, any manifold which admits a cell structure with no one-dimensional cells is simply connected. For instance, complex and quaternionic projective spaces, spheres of dimension at least 2, products of spheres where each has dimension at least 2, and so on.

Given a compact connected orientable manifold with finite fundamental group, we can pass to the universal cover of the manifold – that again is compact, connected and orientable. Conversely, starting with a compact simply connected manifold, we may be interested in what manifolds arise as quotients of this manifold by properly discontinuous actions. The quotient manifold is orientable iff the action is orientation-preserving.

For instance, for an odd-dimensional sphere, the antipodal map is orientation-preserving, so the quotient, which is real projective space, is orientable. For even-dimensional spheres the quotient is non-orientable.

Thus, when trying to classify all compact connected orientable manifolds with *finite* fundamental group, we can proceed as follows:

- First, classify all compact simply connected manifolds
- For each compact simply connected manifold, study the possible properly discontinuous group actions on it, which preserve orientation. The quotients give new compact connected orientable manifolds

For odd-dimensional spheres, the antipodal map is orientation-preserving, while for even-dimensional spheres, it is orientation-reversing. The Lefschetz fixed-point theorem tells us something better: there does not exist a fixed-point free map from the even-dimensional sphere to itself, that is orientation-preserving.

Theorem 10. Any orientation-preserving self-homeomorphism of an even-dimensional sphere, must have a fixed point.

Proof. Since the map is orientation-preserving, the trace on top dimension is 1, so the Lefschetz number is 2. Applying the Lefschetz fixed-point theorem, we see that there is a fixed point. □

In some sense, all the manifolds which have the same universal cover could be studied "together" because they share common properties (for instance, their higher homotopy groups are all equal). However, they may be very different homologically.

8.4. **Compact connected orientables: quick review.** We have seen that for compact connected orientable manifolds, there are the following properties of importance. Most of the properties listed here are properties that embody greater transitivity, homogeneity, and freedom of motion. The opposites of these (like having the fixed-point property, not possessing an orientation-reversing homeomorphism, being even-dimensional) embody greater rigidity and inflexibility:

- Being a compact connected (nontrivial) Lie group
- Being parallelizable
- Having Euler characteristic zero
- Being odd-dimensional
- Having Euler characteristic zero
- Possessing fixed-point-free self-maps
- Possessing orientation-reversing homeomorphisms
- Possessing self-maps of every possible degree
- Being simply connected

Here are the implications among them:

- Compact connected Lie group $\implies$ possesses fixed-point-free self-maps (in fact possesses fixed-point-free self-maps homotopic to the identity)
- Compact connected Lie group $\implies$ possesses orientation-reversing homeomorphisms
- Compact connected Lie group $\implies$ has Euler characteristic zero
- Compact connected Lie group $\implies$ parallelizable
- Odd-dimensional $\implies$ has Euler characteristic zero
- Quotient of compact connected Lie group by finite subgroup $\implies$ parallelizable

On the rigid side, we have spaces like S^{2n}, which does not possess a fixed-point-free orientation-preserving homeomorphism, and whose Euler characteristic is not zero. It is also far from parallelizable. The quotient of S^{2n} by the antipodal map is not even orientable.

Even more rigid than S^{2n} are complex projective spaces of *even* complex dimension. Although orientable, these spaces do not possess orientation-reversing homeomorphisms, nor do they possess fixed-point-free self-maps. Note that odd-dimensional complex projective spaces do possess orientation-reversing homeomorphisms and also possess fixed-point-free self-maps; in fact they possess maps which are both fixed-point-free and orientation-reversing.

9. Constructing manifolds upto homotopy

9.1. **A quick sketch for 3-manifolds.** Assuming that every compact connected orientable manifold can be given a CW-complex structure, we would like to classify all compact connected orientable manifolds upto homotopy using what we know of the theory of CW-complexes. Let us first concentrate on the problem of classifying compact *simply connected* manifolds, for which we already have a headstart for applying Hurewicz's theorem. The idea:

- Classify all compact simply connected manifolds upto homotopy
- Using the classification upto homotopy, find all compact simply connected manifolds upto homeomorphism

- Study all possible properly discontinuous orientation-preserving group actions on these, and thus find all compact connected orientable manifolds with finite fundamental group.

For a compact simply connected manifold M, $H_1(M) = H^1(M) = 0$ and thus, by Poincare duality, $H_{n-1}(M) = H^{n-1}(M) = 0$. We've almost completed the classification of compact simply connected manifolds upto homotopy:

Theorem 11. Any compact simply connected 3-manifold is homotopy-equivalent to the 3-sphere.

Proof. Let M be a compact simply connected 3-manifold. M is orientable, and above remarks show that $H_0(M) = H_3(M) = \mathbb{Z}$ and all $H_1(M) = H_2(M) = 0$. Since M is simply connected, Hurewicz's theorem tells us that $\pi_3(M) = \mathbb{Z}$. Pick a generator for $\pi_3(M)$, and consider a representative map $(S^3, p) \to (M, x)$. This map induces isomorphism on all homologies, and by a combination of Whitehead's and Hurewicz's theorem, is a homotopy equivalence. □

It is acutally true that any compact simply connected 3-manifold is homeomorphic to the 3-sphere, but the proof of this is well beyond the scope of this write-up (it has everything to do with the "Poincare conjecture").

We also get compact connected orientable manifolds with finite fundamental group as quotients of S^3 by finite subgroups. It is not clear if all the properly discontinuous actions are actually realized by subgroups of S^3, but at any rate we get a reasonable number of compact connected orientable manifolds with finite fundamental group.

In contrast, compact connected orientable manifolds like $S^1 \times S^1 \times S^1$ and $S^1 \times S^2$ elude this classification because they have infinite fundamental groups.

9.2. **Generalities.** Given a simply connected topological space, we can construct a CW-complex with a map from the CW-complex to the topological space, which induces isomorphism on all homologies, and is hence a weak homotopy equivalence by Whitehead's plus Hurewicz's. Suppose M is the starting topological space. The idea is to, at the k^{th} stage, choose the attaching maps in such a way as to "kill" the relative homotop between M and the k-skeleton (precisely, we need to take mapping cylinders at each stage).

The construction involves keeping track of two things:

- We need to, at each stage, kill the homology at that stage.
- If the homologies are not free Abelian, then we cannot achieve the required k^{th} homology in the k-skeleton itself – we need to go one skeleton further.

When all the homologies are free Abelian; we do not have the second headache – essentially, there is no interaction between the k-cells and the $k-1$-cells (the boundary of the k-cells can be chosen inside the $k-2$-cells). When all the nonzero homology groups are spaced at a distance of at least 2, we again have no problem, because at each stage we are either killing the "current" homology, or destroying the "carry" from the previous stage. However, when we have torsion *as well as* homology groups in adjacent dimensions, we have to choose attaching maps to take care of both factors.

In all the examples we have seen so far, we do not simultaneously have both headaches. For spheres and complex projective spaces, we have free Abelian homology. In contrast, for real projective spaces, we have torsion in the homology but we have empty homologies in between which we can use to destroy the "carries".

9.3. **Ways of pasting.** Given a specific topological space, the CW-complex which we construct for it is unique upto homotopy equivalence. However, if we are *only* given the sequence of homology groups of the topological space, there may be many possible homotopy types of CW-complexes. The homology groups essentially only measure the interaction between the k-cells and the $k-1$-cells; they do not capture any information about how the attaching map behaves on the $k-2$-skeleton.

Thus, if we want to classify all homotopy types of CW-complexes with a certain sequence of homotopy groups, we need to classify all possibilities for the attaching maps.

For instance, suppose we know the $k-1$-skeleton of a simply connected CW-complex with exactly one k-cell. We want to find what are the possible ways in which the k-cell can be attached. Attaching a k-cell is equivalent to choosing a map from S^{k-1} to the $k-1$-skeleton, and such maps are classified upto homotopy by $\pi_{k-1}(X^{k-1})$. Every element of this homotopy group gives a different possibility for the CW-complex.

If we have multiple k-cells, then we are effectively choosing subsets of $\pi_{k-1}(X^{k-1})$ (the homotopy classes of the attaching maps) and things become more complicated.

9.4. **For 4-manifolds.** Let us now classify all compact simply connected 4-manifolds upto homotopy. Let M be a compact simply connected 4-manifold. Then $H_1(M) = H^1(M) = 0$ and by Poincare duality, $H_3(M) = H^3(M) = 0$. Further $H_2(M) \cong H^2(M)$, and $H^2(M)$ is free Abelian (because $H_1(M) = 0$), hence $H_2(M)$ is free Abelian. Moreover, $H_2(M)$ is finitely generated. Let k be the number of generators.

In the construction of the CW-complex, the 2-skeleton is a wedge of k copies of S^2. Denote this wedge as X^2. There is exactly one 4-cell, and the number of ways we can paste the 4-cell is parametrized by elements of the group $\pi_3(X^2)$. Thus, we see that the third homotopy group of a wedge of spheres controls the possible homotopy types of CW-complexes.

Of course, not all these homotopy types may be equivalent to compact connected orientable manifolds.

Let us consider a few particular cases of this. Suppose $b_2 = k = 1$, viz., $H_2(M) = \mathbb{Z}$. Then the ways of attaching the 4-cell are governed by elements of $\pi_3(S^2)$. It turns out that $\pi_3(S^2) = \mathbb{Z}$, and a generator for this is the Hopf fibration $S^3 \to S^2 = \mathbb{CP}^1$.

Some observations:

- Choosing the generator $1 \in \pi_3(S^2)$, namely the Hopf map itself, yields $\mathbb{CP}^2$.
- Choosing the element $0 \in \pi_3(S^2)$ yields $S^2 \vee S^4$. This is not homotopy-equivalent to a compact connected orientable manifold.
- I believe that choosing other elements in $\pi_3(S^2)$ also gives a complex projective plane, but I'm not sure how one would prove that.

Let's now consider the case $b_2 = 2$. The 3-skeleton is now $S^2 \vee S^2$, and the maps are parametrized by elements of $\pi_3(S^2 \vee S^2)$. Although we do not explicitly know what this group is, we can still pick some elements from it:

- Picking the 0 element here gives $S^2 \vee S^2 \vee S^4$. This is *not* homotopy-equivalent to a compact connected orientable manifold, because the cap product here is trivial.
- We can pick the Hopf map from S^3 to one of the S^2s – this yields $\mathbb{CP}^2 \vee S^2$. This is again not homotopy-equivalent to a compact connected orientable manifold, because the cap product of the S^2-piece with thetop cohomology is zero.
- We can pick an element of $\pi_3(S^2 \vee S^2)$ which corresponds to the boundary of $S^2 \times S^2$, and the CW-complex we get is $S^2 \times S^2$.
- We can first consider the pinch map $S^3 \to S^3 \vee S^3$ and then apply the Hopf fibration on each piece. This gives a connected sum of two copies of $\mathbb{CP}^2$. The relative orientations we choose for the two pieces determine which connected sum we do get.

  Thus, the *wedge* in the 2-skeleton corresponds to the *connected sum* as far as the manifolds are concerned. This is because the "point" at which the wedging is done blows up to a S^3 after we paste the 4-cell.

In general, an element of $\pi_3(X^2)$ can give rise to a compact connected orientable manifold only if it maps essentially into each of the 2-cells; if it can be made to miss even one 2-cells, Poincare duality is contradicted.

As we increase b_2, we get various possible connected sums of $\mathbb{CP}^2$ and $S^2 \times S^2$, by choosing combinations of pinching, the Hopf map, and the map from S^3 to $S^2 \vee S^2$ for pasting $S^2 \times S^2$.

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