CO-INCIDENCE PROBLEMS AND METHODS

VIPUL NAIK

Abstract. Proving concurrence of lines, collinearity of points and concyclicity of points are important class of problems in elementary geometry. In this article, we use an abstraction from higher geometry to unify these classes of problems. We build upon the abstraction to develop strategies suitable for solving problems of collinearity, concurrence and concyclicity. The article is suited for high school students interested in Olympiad math. The first section is meant as a warmup for junior students before plunging into the main problem.

1. Prebeginnings

1.1. **Have we seen concurrence?** Three or more lines are concurrent if they pass through a common point. Where do we see results saying that ... are concurrent?

After having got over the basic definitions in geometry, one of the first substantial results we come across is that the **medians** of a triangle concur. A median is a line segment that joins a vertex of the triangle with the midpoint of the opposite side. The point of concurrence is called the **centroid**.

The beauty doesn't stop there. There are many such *triples* of lines related to the triangle which concur. Let us recall the names of the points of concurrence and the proofs of concurrence of the following :

(1) The **altitudes** – perpendicular dropped from a vertex to an opposite side.
(2) The **perpendicular bisectors** – perpendicular to a side, passing through its midpoint.
(3) The **internal angle bisectors** – line bisecting the internal angle at a vertex.

This article develops tools that help us examine the problem of concurrence in its generality. In subsection 4.4 we shall find that many of these triangle center problems can be solved by just one of the methods used to handle concurrence problems.

1.2. **Have we seen collinearity?** Concurrence means that more than 2 lines have a common point, and collinearity means that more than 2 points are on a common line. Considering that we have so many nice examples of concurrence of lines in **triangle geometry** it is natural to look for similar examples involving collinearity.

Indeed, there are many exciting results and theorems on collinearity, some of which we shall see in this article.

In this article, we will address the question – *how* do we go about showing concurrence? And collinearity? And how are the two problems related?

1.3. **Have we seen concyclicity?** Four or more points are said to be **concyclic** provided that there is a circle passing through all of them. Problems of concyclic points arise as soon as we commence a serious study of the circle. We typically examine them with *angle chasing* tools – showing that the opposite angles are supplementary or that the angles subtended by two points at the other two are equal.

1.4. **What we will do here.** We will knit together all the problem classes described above (collinearity, concurrence and concyclicity) and evolve a unified theme to handle them. And thus, evolve strategies to help us discover as well as prove new and exciting results in geometry.

To do this, at times a little standing back and a little abstraction will be needed. When new terms are introduced it is to make life simpler, and understand the essence of what we are doing. This helps us to get a grip on the fundamental problem, define it more accurately, and solve specific problems better.

In case of new terms whose explanation in the text seems inadequate, refer to the definitions given at the end.

2. The core problem

2.1. **A little motivation.** Three fundamental problem classes in geometry are :

(1) Given more than 2 points, are they **collinear**?
(2) Given more than 2 lines, are they **concurrent**?
(3) Given more than 3 points, are they **concyclic**?

The problem classes have uncannily similar statements. The *best* way of *capturing* that similarity is to introduce an abstract concept of **geometry**. This concept is used in higher math contexts, but is equally useful here.

2.2. **Geometries, varieties, types and incidence.** Problems of collinearity and concurrence tackle relationships between points and lines. Problems of concyclicity tackle relationships between points and circles.

Suppose we are interested only in points and lines in the Euclidean plane. Given a point and a line, there are two possibilities – the point is *incident* on the line (that is, lies on the line) or the point is not incident on the line. **Incidence** is thus a relation (in the mathematical sense) between points and lines.

This inspired mathematicians to define geometry in terms of relations between geometric entities, which they termed **varieties**. The setup is as follows :[1]

- A **variety** is *any* geometric entity being studied.
  In our case (only points and lines), *every* point is a variety, and *every* line is a variety.
- A **type function** gives the type of a variety.
  In our case (only points and lines), there are two types – points and lines. The type function is a map to the set { Point,Line } which takes every point to Point and every line to Line.
- An **incidence relation** which is reflexive and symmetric, such that two varieties of the same types can be incident iff[2] they are identical.
  In our case (only points and lines), a point and a line are incident iff the point lies on the line. Each point is incident on itself, each line is incident on itself. No two distinct points and incident on each other, and no two distinct lines are incident on each other.

[1] This way of defining or describing a geometry extends in many directions. It can be extended to higher dimensional affine spaces, where we consider all the linear varieties. It can be extended to spaces over arbitrary fields. When the set of varieties is finite, then the above becomes a study of finite geometries, and this is an area of active research, mixing combinatorial, geometric and algebraic flavors. It is of importance in understanding finite simple groups, among other things.

[2] iff is standard notation for if and only if

2.3. **Formulation of the problem.** We now need to describe the notions of collinearity, concurrence and concyclicity using the above view of a geometry. Collinearity and concurrence are directly related to the incidence geometry of only points and lines, described above. To understand how, we observe that :

(1) A line is completely determined by any *two* points incident on it. Conversely, given any two points there is a unique line incident on both of them.
(2) A point is completely determined by any *two* lines incident on it. The converse breaks down because of parallel lines, but is almost true – given any two lines that are not parallel, there is a unique point incident on both of them.[3]

We can now see the common pattern. To simplify discussions within the article, we define the concept of **incidence number** (*my own terminology*). The incidence number of one type on the other is the number of varieties of the first type needed to uniquely define a variety of the second type incident on all of them. The two statements given above now translate to :

(1) The incidence number of points on lines is 2 – in standard math, a line is completely determined by 2 points incident on it, and no fewer.
(2) The incidence number of lines on points is 2, barring the case of parallel lines – in standard math, a point is completely determine by any 2 lines incident on it, and no fewer.

We look back at the first two problem classes :

(1) Given more than 2 points, are they **collinear**?
(2) Given more than 2 lines, are they **concurrent**?

We now write out the core problem in its abstract form :

Core Problem 1. *If we are given more varieties of a type than its incidence number on another type, determine whether or not there is a variety of the other type simultaneously incident (or* **co-incident***) on all of them?*

Thus, collinearity and concurrence are special cases of Core Problem 1. We shall use the term **co-incidence problems** (*my own terminology*) for problems that arise as instances of Core Problem 1. In the next subsection we explore whether this core problem covers the third problem class mentioned at the outset – concyclicity.

2.4. **Points and circles.** We work in the Euclidean plane with points and circles as the only types. The incidence relationship is defined as it was for points and lines : a point and a circle are incident iff the point lies on the circle. By the definition of a geometry, two points are incident iff they are the same, and two circles are incident iff they are the same.

We assume for this purpose that a circle has nonzero radius, and lines are treated as circles. The incidence number of points on circles is three, that is, given any three points, there is a unique circle through them.

However, there is no clear cut concept of the incidence number of circles on points. Given any two circles, they may intersect in two points, one point or zero points, so it is not possible to unambiguously define a *unique* point of intersection.[4]

---

[3] If we move to the projective plane $\mathbb{R}P^2$ instead of the affine plane, then the duality of points and lines becomes proper, in the sense that any two lines meet at a point.

[4] In the complex plane, two circles (not both of which are lines) always meet in two points, counting multiplicities. In the complex projective plane two circles (neither of which are lines) will meet at four points – two of them being points at infinity. More generally, in the complex projective plane, the number

Thus, the core problem discussed above cannot handle notions of common intersection points of circles or more general curves. Nonetheless, these notions are important. We shall observe two things :

- The problem of determining whether a collection of circles (or other curves) has common points does not strictly lie within the domain of Core Problem 1. However, many of the heuristics and strategies developed for Core Problem 1 continue to be applicable for such problems.
- There are core problem formulations (Core Problems 2 and 3) that are fundamentally better suited to model problems not well covered by Core Problem 1.

The next section (Section 3) discusses the problem of points co-incident on a variety. This lies strictly within the scope of Core Problem 1 and includes collinearity as well as concyclicity. The section after that (Section 4) discusses the problem of varieties meeting at a point. The problem of concurrence of lines falls within this. This also tackles the more general notion of common intersection points of circles (and more general curves).

2.5. **Lines and circles.** (*This can be skipped!*) Every curve can be treated as the locus of a point. Smooth curves can also be treated as **envelopes** of lines. An **envelope** of a collection of lines is the curve whose **tangents** (refer the definitions at the end) are those lines. For instance, in physics, the electric field lines are the envelopes of the electric field vectors at all points.

We had called a point incident on a curve if it was *on* the curve. A line is thought of as being incident if it *touches* the curve. Using tangency as the incidence relation, we can define a geometry with the two types – lines and circles.[5]

Tangency can be treated as the notion of incidence even between the varieties of an arbitrary collection of types, because the notion of tangency makes sense for any two arbitrary curves.

3. Points co-incident on a variety

3.1. **Some initial observations.** We begin by studying the problem of whether a given set of points is co-incident on some variety of the given type. Here are some examples, of which the first two have already been mentioned as problem classes :

- **Lines** where the incidence number is 2, and the notion of being commonly incident is termed being **collinear**.
- **Circles** where the incidence number is 3 and the notion of being commonly incident is termed being **concyclic**.
- **Conic sections**[6] where the incidence number is 5. The question in this case becomes – given 6 points in the Euclidean plane, is there a conic section incident on all of them?

Observation 1 (Stating the obvious). *Let E and F be two sets of varieties of type ϕ, such that all members of E are incident on a variety of type τ, and all members of F are also incident on a variety of type τ. If $|E \cap F|$ is at least as great as the incidence*

---

of points where a curve of degree m intersects a curve of degree n is mn, counting multiplicities. This is the famous **Bezout's Theorem**.

[5]In the point-circle geometry, lines were allowed to be circles, but points were not. In the line-circle geometry, points are allowed to be circles, but lines are not

[6]Justification of the number 5 can be given as follows – the equation of a general conic has 5 freely varying parameters, and hence 5 points on the conic will give 5 equations that will then determine the conic

number of ϕ on τ, then the two varieties coincide, so $E \cup F$ has all its members incident on a variety of type τ.

This observation is crucial in guiding our thinking and strategies. We shall come across it repeatedly while solving problems and developing tools for solving co-incidence problems. For this section, ϕ will usually be the variety of points.

3.2. **Finding the variety.**

Heuristic 1 (Finding the variety)**.** *To prove that there is a variety incident on all of a given collection of points, we try to* find *a variety and then* show *that every point (in the collection) is incident on it. Finding the variety (line, circle or conic section) often amounts to expressing it as a locus of some condition satisfied by all the points.*

We look at an illustration of this method in a problem of proving *eight* points to be concyclic. We *find* the circle in question and then *show* that each of the points lies on this circle. For definitions of tangents refer the glossary.

Problem 1 (Concyclicity of eight points)**.** *Consider two circles with non-overlapping interiors. Take the four points of intersection between the direct and indirect common tangents to these two circles, and the four points of contact for tangents from the center of each circle to the other one. Then* **show** *that these eight points are concyclic.*

Proof. If O_1 and O_2 are the centers of the two circles, we can show that all the points subtend an angle of $\pi/2$ at O_1O_2. The corresponding locus is the circle with diameter O_1O_2. Hence we conclude that all the eight points are concyclic. □

3.3. **Methods of elimination and angle chasing.** Suppose we have to show that there exists a line through three given points. The statement of *existence* of the line can often be converted to a statement that purely describes the *relationship* between the points. This relationship could be described in geometric terms, or by means of coordinate geometry or complex numbers. We shall restrict to the geometric interpretation for this article.

In geometric terms: The angle made at a point between the line segments joining it to the other two points is 0 or π. If the three points are A, B and C, then $\angle BAC = 0$ or π. It is π when A is between B and C and 0 otherwise.

In complex number terms: (*We won't use it in the article!*)If z_1, z_2 and z_3 are the affixes of the three points, then $\frac{z_3-z_1}{z_2-z_1}$ is real. Interpreted in terms of arguments this gives the previous description.

In coordinate geometry terms: (*We won't use it in the article!*)Let (x_1, y_1), (x_2, y_2) and (x_3, y_3) be the coordinates of the three points. Then the *slope* of the line joining (x_1, y_1) and (x_2, y_2) is the same as the slope of the line joining (x_1, y_1) and (x_3, y_3). Equivalently

$$\frac{y_2 - y_1}{x_2 - x_1} = \frac{y_3 - y_1}{x_3 - x_1}$$

In case $x_1 = x_2$ (so the slope is not defined) the condition is that $x_1 = x_3$ as well. Another way of putting this is that the determinant of a certain matrix is 0.

To prove that more than 3 points are concyclic, we have similar conditions :

In geometric terms: Let A, B, C, D be the four points. If A and B are on the same side of the line CD then they must subtend equal angles at CD (that is,

$\angle CAD = \angle CBD$). If A and B are on opposite sides of CD then they must subtend supplementary angles at CD (that is $\angle CAD + \angle CBD = \pi$) [7]

In complex number terms: (*We won't use it in the article!*)Let z_1, z_2, z_3 and z_4 be the four points. If any three of them are collinear, the fourth also must lie on the same line. Otherwise $\arg \frac{z_1 - z_4}{z_1 - z_3} = \arg \frac{z_2 - z_3}{z_2 - z_4}$.

In co-ordinate geometry terms: (*We won't use it in the article!*)The geometric condition is translated to a relation among slopes.

Key Point 1. *The* existence *of an element satisfying certain properties is converted to the problem of verifying some* relation *between the entities and quantities known to us.*

We may already have com across some examples of this in algebra. For instance the statement of existence of a root in $\mathbb{R}$[8] for a quadratic equation with coefficients in $\mathbb{R}$ can be converted to a statement that the **discriminant** of the quadratic equation is nonnegative. Similarly, the existence of a solution to a system of two linear equations can be converted to a system of conditions involving determinants.[9]

Once the problem has been reduced to establishing some relation between the knowns, then we can resort to elementary **diagram chasing**. In particular when the relations involve sums and differences of angles (as is true for the methods described above), we use the term **angle chasing**. Some results on collinearity and concyclicity proved via angle chasing :

(1) **Simson's Line Theorem** states that the projections[10] from a point on the circumcircle of a triangle to its sides are collinear. The line is termed the **Simson's Line** or the **Simson-Wallace Line**. The converse is also true and can be proved from the theorem.
(2) **Miquel's Theorem**[11] states that if $ABCD$ is cyclic and P, Q, R and S are points such that A, B, P, Q are concyclic, B, C, Q, R are concyclic, C, D, R, S are concyclic, and D, A, S, P are concyclic, then P, Q, R, S are also concyclic.

3.4. **Ratio methods and other criteria.** The above methods for elimination (both in the case of collinearity and concyclicity) involved a direct interpretation in terms of angles. There are cases where collinearity of points obtained by some procedure can be translated to conditions involving other parts of the diagram in an unexpected way. Here we discuss some powerful results that help us convert a problem of collinearity into a problem of determining relations between known quantities :

(1) **Menelaus' Theorem** : Given a triangle $\triangle ABC$, and points D, E, F on sides BC, CA and AB respectively, the product of the **signed ratios**[12] BD/DC, CE/EA and AF/FB is -1 iff the points D, E and F are collinear.

---

[7] These multiple cases often arise when using synthetic or visual tools. Sensed angles can be used to club the two cases

[8] $\mathbb{R}$ denotes the set of reals

[9] In model theory, the procedure of removing the $\exists$ and $\forall$ quantifiers from statements is termed **quantifier elimination**, and there are some theories that admit quantifier elimination for every statement! In fact the theory of **real closed fields**, for which $\mathbb{R}$ is the prototypical example, admits quantifier elimination, as was seen in the example of existence of roots for a quadratic equation.

[10] projection of a point on a line or plane is the foot of the perpendicular from the point to the line or plane

[11] There are many other results discovered by and named after Miquel, so this theorem may not be the same as a Miquel's theorem stated elsewhere

[12] A signed ratio is a ratio of signed lengths on the same line, the sign being given to a length based on an arbitrary choice of direction. The sign of the ratio is independent of the choice of direction

(2) **Desargues' Theorem** states that if $\triangle ABC$ and $\triangle A'B'C'$ are two triangles then AA', BB' and CC' concur iff $AB \cap A'B'$, $BC \cap B'C'$ and $CA \cap C'A'$ are collinear. If either of the equivalent conditions hold, the two triangles are said to be **in perspective** or **perspective triangles**.
This helps us switch between collinearity of different sets of points.

(3) A variant of **Bezout's Theorem** (*This can be skipped!*): A cubic is a curve in the plane that represents the locus of a degree 3 relationship between x and y in a Cartesian coordinate system. If two cubics intersect in nine points and we choose three of them then those three points are collinear if and only if one of these hold : both the cubics contain the line through those three points, or the remaining six lie on a conic section (a curve that represents a degree 2 relationship).
This helps us switch between a collinearity problem and a problem of points lying on a conic.

(4) **Pascal's Theorem** (*This can be skipped!*): If A, B, C, D, E, and F are six points then $AB \cap DE$, $BC \cap EF$ and $CD \cap FA$ are collinear iff A, B, C, D, E, F are on a conic. This can be deduced from Bezout's Theorem.
This helps us switch between a collinearity problem and a problem of points lying on a conic.

(5) **Pappus Theorem** (*This can be skipped!*): A special case of Pascal's Theorem where A, C and E are given to be collinear. The theorem now becomes – $AB \cap DE$, $BC \cap EF$ and $CD \cap FA$ are collinear iff B, D and F are collinear.

3.5. **Physical insights into problems.**

Problem 2 (Monge's Theorem). *The pairwise external centers of similitude*[13] *for three circles (with pairwise unequal radii) are collinear.*

The result follows directly from Menelaus' Theorem, applied to the triangle with vertices being the centers of the three circles.

There is also an interesting physical interpretation in three dimensions : Consider three balls of not necessarily equal sizes resting on a table and consider a flat planar sheet placed on top of all three. The table and the sheet are the direct common tangent planes to the three spheres, and the line where the sheet meets the table, is precisely the line of interest.

Concept Testers

(1) Given circles Γ_1, Γ_2 and Γ_3 with distinct radii, let P_{12} be the internal center of similitude for Γ_1 and Γ_2, P_{23} be the internal center of similitude for Γ_2 and Γ_3, and P_{31} be the *external* center of similitude for Γ_1 and Γ_3. **Prove** that P_{12}, P_{23} and P_{31} are **collinear**.
(2) If A,B, C and D are concyclic and E, F are points on AB and CD respectively such that $EF \parallel BC$, **prove** that A, B, E and F are **concyclic**.
(3) Let $\triangle ABC$ be a triangle. Let D, E and F be the points where the external angle bisectors of angles A, B and C meet the opposite sides. Prove that D, E and F are collinear.

[13]an external (internal) center of similitude is a point dividing the line joining the centers of the circle externally (internally) in the ratio of their radii. The internal center of similitude is the point of intersection of indirect common tangents (if they exist) and the external center of similitude is the point of intersection of direct common tangents (if they exist). See terms and definitions at the end for details.

4. Varieties meeting at a point

4.1. **An improved formulation.** We are interested in learning about the common intersection points of a collection of varieties. One possibility is that all the varieties are **concurrent** – that is, there is a point incident to all of them. Such a point is termed a **point of concurrence**.

As discussed earlier (subsection 2.4), this notion has shortcomings. A collection of circles may have 1 common intersection point, or 2 common intersection points. Even in the case of lines, parallel lines have no common intersection points.

The main utility of Core Problem 1 lay in the fact that problems in the classes of collinearity, concurrence, and concyclicity could be expressed as instances of this. In the same way, we seek a new core problem formulation such that problems of common intersection points of varieties become special cases of it.

Such a formulation is given below :

Core Problem 2. *For a given collection of varieties, determine whether or not the following condition holds : there is a fixed set of points S such that given any two varieties in the collection, their set of intersection points is S.*

For instance, if the varieties we are handling are lines, and two of them intersect, then all the lines must pass through the point of intersection. On the other hand, if two of them are parallel, then all the lines must be parallel to them.

4.2. **Algebraic reformulation.** (*This can be skipped!*) In coordinate geometry, a set of points is treated as a locus of certain relationships expressed between the coordinates. For instance, the unit circle centered at the origin is represented by the equation $x^2 + y^2 = 1$, or better still, $x^2 + y^2 - 1 = 0$.

If two curves have equation $F_1 = 0$ and $F_2 = 0$ where F_1 and F_2 are both expressions in terms of x and y, then their intersection points must satisfy both $F_1 = 0$ and $F_2 = 0$. Thus, those intersection points also lie on all curves of the form $G_1F_1 + G_2F_2 = 0$ where G_1 and G_2 could be arbitrary expressions. This collection of expressions is said to be the **ideal** generated by the two given expressions. An ideal such that the n^{th} root of any element in it also lies in it is a **radical ideal**.

Not much can be said about ideals in general. However, when the expressions are restricted to *algebraic expressions* or *polynomial expressions* then the corresponding varieties are termed **algebraic varieties** and their structure is extremely well behaved. Lines, circles and conic sections are all examples of algebraic varieties, but the spiral and sine wave are not.

A powerful result in algebraic geometry, namely the **Hilbert's nullstellensatz**, says that the *only* algebraic varieties passing through the intersection points of certain algebraic varieties are in the radical ideal generated by them, when working over complex numbers.

This suggests that a more fundamental equivalent of 2 is the following :

Core Problem 3. *Given a collection of algebraic varieties, determine whether or not the following condition holds : there is an ideal I such that I is the radical ideal generated by any two member varieties.*

The two formulations are the same when working over complex numbers. Over real numbers, Core Problem 3 checks a stronger condition than Core Problem 2.[14]

[14] However, Core Problem 3 has the disadvantage that it makes sense only for algebraic varieties.

A simple case when both formulations are equivalent even over the real Euclidean plane, is that of lines. Given a collection of lines, they satisfy the condition of Core Problem 2 if and only if given any two lines, every other line in the collection can be obtained as a linear combination of those two lines. This latter statement corresponds to the condition of Core Problem 3.

4.3. **Coaxial circles.** The **radical axis** of two circles is a straight line giving the locus of a point having equal powers with respect to the two circles[15] There are three cases :

- The two circles touch (internally or externally) : In this case the radical axis is the tangent through the common point.
- The two circles meet in two points : In this case the radical axis is the common chord.
- The two circles do not intersect : In this case the radical axis intersects neither.

A collection of circles is termed **coaxial** provided that the radical axis of any two member circles is the same.

When a collection of circles is coaxial, the points of intersection of any two members must lie on all the other members. This satisfies the condition being checked in Core Problem 2.

A **coaxial system** is a collection of coaxial circles that cannot be made bigger.

Coaxial systems are of three types :

- The **intersection point type** where every pair of circles intersects at two points. The two intersection points thus define the radical axis.
- The **limit point type** where no two circles intersect.
- The **tangent type** where every pair of circles is tangent.[16]

Coaxiality is the proper notion corresponding to Core Problem 2. [17]

For those who have read the previous subsection, coaxiality also corresponds to Core Problem 3. In an early co-ordinate geometry treatment of circles, we come across families of circles parameterized as $S_1 + kS_2$ where S_1 and S_2 are the equations of two circles. This is a parameterization of the coaxial system of circles containing both S_1 and S_2. The parameterization shows that every circle in the coaxial system containing two circles is a linear combination of their equations.

The concept of coaxiality transforms a problem looking for common *intersection points* to looking for a common *radical axis.*

Heuristic 2. *To prove that a collection of circles is concurrent, it suffices to show that they are coaxial, and that at least two of them intersect. To show that the circles are coaxial, it suffices to show that the radical axis of any two circles has equal powers with respect to all the circles.*

4.4. **Symmetric formulations.** We consider the following heuristic for proving that given any three curves C_1, C_2 and C_3 their pairwise intersections are the same (that is, Core Problem 2):

Heuristic 3 (Symmetric condition)**.** *Construct a symmetric condition and show that it is equivalent to the condition of being an intersection of any two of the curves.*

---

[15] The power of a point with respect to a circle is given by $d^2 - r^2$, where d is its distance from the center, and r is the radius.

[16] Suitable inversion take the intersection point type to a system of concurrent lines, the limit point type to a system of concyclic circles, and the tangent type to a system of parallel lines. These are degenerate cases of coaxial families.

[17] Remains so even on moving to $\mathbb{C}^2$

We often proceed to do this by finding functions f_1, f_2 and f_3 from the points on the plane to $\mathbb{R}$ such that C_1, C_2 and C_3 are expressed as loci as follows :

$$\begin{aligned} C_1 &\equiv (f_2(P) = f_3(P)) &\equiv (f_2(P) - f_3(P) = 0) \\ C_2 &\equiv (f_3(P) = f_1(P)) &\equiv (f_3(P) - f_1(P) = 0) \\ C_3 &\equiv (f_1(P) = f_2(P)) &\equiv (f_1(P) - f_2(P) = 0) \end{aligned}$$

Thus, C_1 is the locus of a point at which f_2 and f_3 take the same value. C_2 is the locus of the point at which f_3 and f_1 take the same value. C_3 is the locus of the point at which f_1 and $f_@$ take the same value.

The corresponding symmetric condition becomes :

$$f_1(P) = f_2(P) = f_3(P)$$

This is equivalent to any two of the three equations being true. Thus, the set of intersection points of any two of the three curves is the set of points satisfying this condition.

We look at some typical examples from triangle geometry :

(1) **Perpendicular bisectors** of sides of a triangle correspond to the f_is being distances from the vertices. They concur at the **circumcenter**. More generally, all perpendicular bisectors concur for any system of concyclic points.
(2) **Angle bisectors** of vertex angles correspond to the f_is being distances from sides. The concurrence points are the **incenter** and **excenters**. More generally all angle bisectors concur where the lines are tangents to a circle (that is, we have a polygon with a circle touching all its sides). Whether the angle bisector taken is the internal or external one depends on the manner in which the circle touches the sides.
(3) **Appollonius circles** of each pair of vertices passing through the third. The functions f_i are $PA.BC$, $PB.CA$, $PC.AB$ (the . represents product of magnitudes, not dot product). The three circles are coaxial, always have real intersection points, and the two points of intersection, known as the **isodynamic points** are inverse points of each other with respect to the circumcenter[18].
(4) **Altitudes** where the f_is are *cyclic permutations* of $PA^2 + BC^2$. That is, the other f_is are obtained by cyclically permuting A, B, C in the expression. They concur at the **orthocenter**.
(5) **Medians** where the f_is are cyclic permutations of $\mathbf{Ar}.\triangle PBC$. They concur at the **centroid**.

Observation 2. *It is often useful in triangle geometry to view cyclic permutations of an expression (or any construct) in terms of the vertices, that is, to view the expressions (or constructs) obtained by cyclically permuting the vertices in the expression (or construct).*

We can similarly prove that given three circles, the pairwise radical axes are concurrent. The functions f_i in this case are the power of the point with respect to the circles C_i.

4.5. **Non-equational symmetric formulations – concurrent circles.** There are cases where we exploit the same kind of idea as above (subsection 4.4) but not in an equational sense. We discuss two problems of concurrent (not coaxial circles) – one involving the **complete quadrilateral** and the other involving the **complete quadrangle**.

[18] This follows from the fact that each Appollonius circle is orthogonal to the circumcircle

4.5.1. *The complete quadrilateral.* A **complete quadrilateral** is a collection of 4 distinct lines, such that no two of them are parallel, and no three of them are concurrent. These 4 lines are *not in any cyclic order*. The lines are termed the **sides** of the complete quadrilateral. A complete quadrilateral has :

- $\binom{4}{2} = 6$ points of intersection of the sides two at a time, known as the **vertices**.
- $\binom{4}{3} = 4$ triangles, formed by taking the sides three at a time.
- 3 diagonals, the line segments formed by joining pairs of **opposite vertices**, that is, vertices that are obtained as intersections of disjoint pairs of sides.

Problem 3 (Existence of Miquel point)**.** *Prove that the circumcircles of the triangles of a complete quadrilateral are concurrent. (The point of concurrence is termed the* **Miquel point***).*

While drawing a diagram for this problem, it is best *not* to make all the four circles. Rather, we know that if there is a common point, then that point can be determined just by intersecting two of the circles. Thus, for the purpose of this problem, it suffices to draw only 2 of the circles.

We first note that every vertex occurs as an intersection point of two of the circles. Thus, if the four circles were to have a common radical axis, every vertex would lie on it. This is not possible. Thus, this is an example where the four circles are concurrent without being coaxial – they have only 1 common intersection point, not two.

Going by the idea used above, viz Heuristic 3, we need to prove that an intersection of two of the circumcircles must lie on the other two circumcircles as well. The difference here : one of the intersection points is a vertex, and this will not lie on the other two circumcircles. It is the other intersection point (that is not a vertex) that lies on all four circumcircles.

Proof. Let l_1, l_2, l_3 and l_4 be the four lines of the complete quadrilateral. Consider the circumcircles of the triangles formed by (l_1, l_2,l_3) and by (l_1,l_2,l_4). These two circles meet at the intersection point of l_1 and l_2. A simple verification shows that if the two circles are tangent at this point, then $l_3 \parallel l_4$ which is not allowed. Thus, the two circumcircles must intersect in another point. Let this be P.

Now we try to apply Heuristic 3, by finding some symmetric condition whose locus is the circumcircle. This symmetric condition has to be a property of that point with respect to the *lines*, because a complete quadrilateral is a set of 4 lines. The **Simson's Line Theorem** and its converse give us that a point is on the circumcircle of the triangle formed by three lines iff its projections on the three lines are collinear.

The projections of P on l_1, l_2 and l_3 are collinear. Similarly, the projections of P on l_1, l_2 and l_4 are collinear. These two collections of collinear points have two points in common. Thus, as per Observation 1 (taking ϕ as points and τ as lines), the projections of P on all sides of the complete quadrilateral are collinear. This in turn gives that P lies on the other two circumcircles as well.

□

We glean some things from the above proof:

- The symmetric condition of Heuristic 3 was not an equality (as in subsection 4.4) but rather a collinearity. Establishing the condition in turn required the use of the principles of collinearity in the form of Observation 1.
- The proof show that the intersection of two circumcircles which is *not* a vertex lies on the other two circumcircles. The intersection which is a vertex is $l_1 \cap l_2$, and its projections on the sides l_1 and l_2 coincide with itself. Thus, we do not

have enough numbers to be able to apply Observation 1. Thus the symmetric condition being actually used is : the projections on the sides are *distinct* and collinear.

4.5.2. *The complete quadrangle.* A **complete quadrangle** is a collection of 4 distinct points, such that no three of them are collinear. These points are *not in any cyclic order*. The points are termed the **vertices** of the complete quadrangle. We have :

- $\binom{4}{2} = 6$ lines formed by the points two at a time, known as the **sides**.
- $\binom{4}{3} = 4$ triangles, formed by taking the points three at a time.

Problem 4 (Concurrence of nine point circles)**.** *Prove that the nine point circles of the triangles of a complete quadrangle are concurrent.*

There is an elementary proof of this by angle chasing (by methods we shall see in the next subsection). Here, we outline a proof based on a symmetric condition, that links this up with conic sections.

Proof. (*This can be skipped!*)The nine point circle of an orthic system is the locus of the center of a rectangular hyperbola passing through the four points of the orthic system (if a rectangular hyperbola passes through three points of an orthic system, it must pass through the fourth).

As in the example involving complete quadrilaterals, the nine point circles of two of the triangles intersect in one midpoint, and another unknown point. That unknown point is the center of a (unique) rectangular hyperbola passing through the vertices of one triangle, and is also the center of a (unique) rectangular hyperbola passing through the vertices of the other triangle.

We now use the fact that a rectangular hyperbola is completely determined by its center and any two points on it that are not opposite to each other. Thus, the incidence number of points on rectangular hyperbolas with a fixed center is 2, subject to the condition that the two points are not opposite.

This lets us apply Observation 1 to conclude that as the rectangular hyperbolas corresponding to the two triangles have two common points, they are identical and thus the given point is the center of a rectangular hyperbola passing through all four points. ☐

We glean some things from the above proof :

- The symmetric condition of Heuristic 3 was not an equality (as in subsection 4.4) but rather the property of being incident on the same rectangular hyperbola. Establishing the condition in turn required the use of Observation 1, after determining the incidence number of points on rectangular hyperbolas with fixed center.
- The proof show that the intersection of two circumcircles which is *not* the midpoint lies on the other two circumcircles. The problem with the intersection which is the midpoint is as follows : the two points common to the two rectangular hyperbolas are opposite to each other with respect to the center.

The parallel between both examples is amazing. We can also work out a conic sections proof of the Miquel point problem. (*This can be skipped!*)That proof rests on the following :

Lemma 1. *The locus of the focus of a parabola tangent to three non-concurrent lines is the circumcircle of the triangle they form, minus the vertices of the triangle. The parabola corresponding to each focus is unique. Moreover, the Simson's line of the focus with respect to the triangle is the tangent through the vertex of the parabola.*

We can use this lemma to perform reasoning identical to the above. At some point, while applying Observation 1 we will need to determine the incidence number of lines on parabolas with a fixed focus. Here, incidence of a line and a parabola is defined as the line being tangent to the parabola (incidence relationships involving lines were discussed in subsection 2.5).

What we finally get is :

Claim 1 (About the Miquel point)**.** *The Miquel point is the focus of the unique parabola tangent to all the four lines. The projections from it to the four sides are collinear and form a line called the* **pedal line***, which is also the tangent through the vertex of the parabola.*

4.6. **Angle totals – concurrence to collinearity.** In the study of methods to prove collinearity, we had stressed on two broad paradigms – *find* the line, and *eliminate* by reducing the problem of existence of a line to some other relationships. The symmetric condition technique is analogous to the *find* paradigm. We now discuss the *eliminate* paradigm for concurrence. The first step typically reduces problems of concurrence of lines to problems of collinearity of points, and problems of concurrence of circles to problems of concyclicity of points.

Observation 3 (Concurrent to collinear, concyclic)**.** *The problem of concurrence of three lines is equivalent to the problem of the intersection of two lines being collinear with points on the third. Similarly, the problem of concurrence of circles is equivalent to the problem of the intersection of two circles being concyclic with points on the third.*

The typical *angular tools* used in this are :

- The total angle around a point is 2π.
- The angle made by a straight line at a point on it is π.
- The angle sum of a triangle is π, that of a quadrilateral is 2π.

The following results on concurrence of circles can be proven by *angle chasing* using the above ideas :

(1) The nine point circles of a complete quadrangle concur – this fact can be proved by *angle chasing* using the above notions. Showing that the pedal circles also concur at the same point requires considerably more angle chasing.
(2) **Pivot theorem** stating that given a triangle ABC with points P, Q, R on BC, CA, AB the circumcircles of QAR, RBP, PCQ concur.
(3) **Triangle reflections theorem** stating that if P is a point in a triangle ABC and D, E, F are its reflections in BC, CA, AB then the circumcircles of triangles EAF, FBD, DCE concur.

4.7. **Ratio methods and other criteria for concurrence.** These can be thought of as the concurrence analogues of results in collinearity such as Menelaus' Theorem. Situations often arise where problems of concurrence can be reduced to computations of ratios or to completely different problems. The commonly used methods are :

(1) **Ceva's theorem** states that if D, E, F are on sides BC, CA, AB of a triangle $\triangle ABC$ then AD, BE, CF concur iff the product of the signed ratios BD/DC, CE/EA, AF/FB is unity. (A signed ratio along a line is a ratio of magnitudes, with the prefixed sign indicating whether the measurements were in the same direction). The three line segments AD, BE, CF are termed *Cevians.* Proving this often involves showing something like $BD/DC = f(B)/f(C)$ so that the product *cyclically* becomes 1.

(2) **Trigonometric form of Ceva's theorem** gives another necessary and sufficient criterion for the concurrence :
$\prod \sin \angle BAD/\sin \angle DAC$ is unity. As before the proof often involves showing that $\sin \angle BAD/\sin \angle DAC = f(C)/f(B)$ so that the product cyclically becomes 1.

Some results that use Ceva's Theorem in one of its forms are :

(1) A variant of **Monge's Theorem** stating that given three circles the lines joining each center to the internal center of similitude of the other two circles concur. This is a straightforward corollary of Ceva's Theorem. Similarly if for two pairs of circles we take external centers of similitude, and for the third pair we take the internal center of similitude, the concurrence result holds.
(2) **Seven circles theorem** stating that if we take a circle and inscribe six circles inside it touching it internally such that they form a chain, then the lines joining opposite points of contact are concurrent. As an intermediate we can first show that $AB.CD.EF = BC.DE.FA$ if A, B, C, D, E, F are the points of contact in cyclic order. This immediately implies the concurrence by the trigonometric form of Ceva's Theorem. The proof of the first part rests on some basic trigonometry, that we do not discuss here.

Apart from the Ceva's Theorem there are some other important results involving relatively more complicated configurations :-

(1) **Brianchon's theorem** stating that AD, BE, CF concur iff we can construct a conic touching AB, BC, CD, DE, EF, FA. In particular, in the above seven circles case a conic can indeed be inscribed in the hexagon (thus it is a **bicentric hexagon** as the six points are already concyclic).
(2) **Desargues' Theorem** that we had encountered earlier on.

4.8. **Ceva's theorem and triangle centers.** We saw earlier in section 4.4 that the Symmetric Condition Heuristic (heuristic 3) was used to prove a number of results related to the concurrence of lines defined for a triangle. Ceva's theorem in its geometric as well as trigonometric form is also very useful for proving concurrence of lines (though it is limited only to lines). Moreover, it can sometimes be used to prove *conditional concurrence* results – results saying that if these three lines are concurrent, so are those three. We shall discuss conditional concurrence problems at a later stage.

Three major concurrence results directly obtained from Ceva's Theorem :

(1) The **medians** divide the opposite side in the ratio 1 : 1. Thus, the cyclic product associated with Ceva's theorem becomes 1, and hence the three medians are concurrent.
(2) The **internal angle bisectors** divide the opposite side in the ratio of the sides containing the angle. For instance, in $\triangle ABC$ the bisector of $\angle A$ divides BC in the ratio AB/AC. Here again, the cyclic product becomes 1, hence the three internal angle bisectors are concurrent. In the same way, we can show that two external angle bisectors are concurrent with the third internal angle bisectors.
(3) The **altitudes** divide the opposite side in the ratio of the tangents of the base angles (those who are unaware of trigonometric ratios can skip this). This product also becomes 1 cyclically.

Concept Testers

(1) Two points P and Q on the line AB are termed **harmonic conjugates** with respect to AB if $AP/PB = -AQ/QB$, that is, the ratio in which P divides AB is the same as the ratio in which Q divides AB (one dividing it internally and

the other dividing it externally). **Prove** that if P and Q are harmonic conjugates with respect to AB then A and B are harmonic conjugates with respect to PQ.

(2) Let $\triangle ABC$ be a triangle with P, Q and R on BC, CA and AB respectively. Suppose AP, BQ and CR are concurrent. Let P', Q' and R' be harmonic conjugates of P, Q and R with respect to the lines BC, CA and AB respectively. Prove that :

  (a) P', Q and R are collinear.
  (b) P, Q' and R are collinear.
  (c) P, Q and R' are collinear.
  (d) P', Q' and R' are collinear.
  (e) AP', BQ' and CR are concurrent.
  (f) AP, BQ' and CR' are concurrent.
  (g) AP', BQ and CR' are concurrent.

(3) Let $\triangle ABC$ be a triangle. Let D be defined as the midpoint between the points where the internal and external bisectors of $\angle A$ meet BC. Analogously, define E for $\angle B$ and F for $\angle C$. **Prove** that D, E and F are **collinear**. (*Hint :* Either directly use Menelaus' Theorem or use the fact that the centers of coaxial circles are collinear).

(4) Let $\triangle ABC$ be a triangle. Define Γ_1 to be the locus of P satisfying $\frac{PB}{CA} = \frac{PC}{AB}$, Γ_2 to be the locus of P satisfying $\frac{PC}{AB} = \frac{PA}{BC}$ and Γ_3 to be the locus of P satisfying $\frac{PA}{BC} = \frac{PB}{CA}$. **Prove** that the curves Γ_1, Γ_2 and Γ_3 satisfy the conditions of **Core Problem 2**.

(5) (*This can be skipped!*) In the previous problem use **Ptolemy's inequality** to establish that the curves Γ_i with $1 \leq i \leq 3$ are concurrent iff $\triangle ABC$ is either acute angled or right angled.

5. Summary and general conclusions

Over the course of this article, we have examined a number of problem classes of coincidence problems. We gave three core problem formulations each of which had certain advantages. We also saw how a general abstraction helps in chalking out specific strategies for solving problems.

An addendum to this article provides some more challenging examples that require us to knit together the various strategies for collinearity, concurrence and concyclicity. It also contains some trickier exercises.

Appendix A. Some illuminative and interesting examples

We now begin looking at some gems of problems that build upon the ideas developed so far.

A.1. Gauss Bodenmiller theorem.

Problem 5 (Gauss Bodenmiller Theorem)**.** *Given a complete quadrilateral, the four orthocenters corresponding to its triangles are collinear.*

There are two proofs of this.

The first uses the theory of conic sections, and in that sense, is not strictly an Olympiad proof. The idea is to determine the *canonical line*, which, in this case, is the directrix of the unique parabola tangent to the four lines. Then, the fact that each orthocenter lies on the directrix picks directly from the fact in conic sections that the orthocenter of the triangle formed by any three tangents to a parabola lies on the directrix.

We shall here discuss an alternative proof, that uses methods of elementary geometry outlined earlier in this article.

Before starting out with a geometric proof, it is advisable to make a rudimentary diagram. As before, we will not clutter the diagram too much by drawing all the altitudes. Rather, we will first focus on the altitudes of just one triangle. If need be, we could construct the altitudes of another triangle. Making more than that is pointless because the line through the orthocenters is determined completely by any two orthocenters.

Proof. To prove that the orthocenters are collinear, we must choose between *finding* the line and eliminating it. The elimination approach using angular tools (subsection 3.3) seems quite forbidding geometrically, and the configuration does not seem amenable to any of the other elimination tools discussed in subsection 3.4. The main hope lies in finding some description of the line and proving that all the orthocenters lie on the line.

Let us examine each triangle more closely. Each triangle contains three vertices, and no two of them are opposite. Thus, for each diagonal, one of its endpoints is a vertex of the triangle. This suggests that we look at some property of the orthocenters that can be define with respect to the diagonals.

If H is the orthocenter of a $\triangle ABC$, and D, E and F are the projections of H on the sides BC, CA, and AB, then

$$AH.HD = BH.HE = CH.HF$$

This property translates to the statement : each orthocenter has equal powers with respect to the circles having diagonals as diameters.

Before proceeding further, we note that the four orthocenters must be distinct points.

The locus of a point having equal powers with respect to any 2 circles is a straight line (namely, the radical axis). This is the line we had sought to *find*. So, the 4 orthocenters lie on the radical axis of any 2 of the circles and are hence collinear.

Also, as 2 points completely determine a line (the incidence number of points on lines being 2) and the four orthocenters are common to the radical axis of every pair among the there circles with diagonals as diameters, we conclude that the three circles are coaxial.

Incidentally this also shows that the midpoints of the diagonals are collinear, because they are centers of coaxial circles. The collinearity of midpoints of diagonals is usually proved via Menelaus' theorem or some similar approach. □

We find that even in examples where none of the approaches we have learnt so far seem to be of direct use, and some leap of thought is required, the approaches we have learnt still play a very important role in guiding our intuition.

A.2. **A selection test problem.**

Problem 6 (Indian IMO Selection Test, 1995)**.** *Let $\triangle ABC$ be a triangle and let D, E be points on AB and AC respectively, such that $DE \parallel BC$. Let P be a point inside $\triangle ADE$ and let F and G be the intersections of DE with the lines BP and CP respectively. Let Q be the second intersection point of the circumcircles of the triangles $\triangle PDG$ and $\triangle PFE$. Prove that A, P, Q are collinear.*

The diagram here needs to be drawn carefully, with pencil and scale, because a mistake in drawing the diagram could make the proof much longer to come by.

Proof. The canonical line in this case is clearly the line PQ which can be interpreted as the radical axis of the circumcircles of triangles PDG and PFE. As P and Q already lie on this radical axis, the problem reduces to showing that A has equal powers with respect to the two circles. This suggests that, if M be the second intersection of the circumcircle of $\triangle PDG$ with AB and N the second intersection of the circumcircle of $\triangle PEF$ with AC, then :

$$AM.AD = AN.AE$$

or, in other words, the points M, D, N and E are concyclic. This is readily verified to being equivalent to saying that M, N, B and C are concyclic. The advantage of going to BC from DE is that the former is, in some sense, more controllable.

Now what quadrilaterals are already known to be cyclic? M, D, G, P are concyclic and thus M, B, C, P are concyclic (for the same reason as above). Similarly N, E, F, P are concyclic and thus so are N, B, C, P.

Here comes the simple but crucial step : using N, B, C, P being concyclic and M, B, C, P being concyclic to show that M, N, B, C are concyclic. The result mentioned right at the beginning of the article (the observation 1) comes in useful here – as the two concyclic sets have 3 points in common, in fact all the 5 points are concyclic and the problem is solved. □

(**Note :** A poorly drawn diagram, in particular one where the alleged circle does not appear to be convex, will hinder this otherwise trivial insight).

A.3. **The Malfatti problem – Ajima Malfatti point.** Malfatti wanted to know how to choose three circles inside a given triangle whose total area was maximum. He presumably considered the problem solved when he discovered that it was always possible to arrange three circles, each touching the other two, and each touching a different pair of adjacent sides.

As it turned out this wasn't true – a better arrangement would be to take the incircle and choose two of the incircles of the three corner curvilinear triangles whose sizes are more. Also when the triangle is long and thin, an arrangement of the circles in a line might be even better. The original solution was finally proven to be never optimal.

The three circles originally considered by Malfatti and now known as **Malfatti circles** have some interesting geometrical properties, one of which we shall try to prove here :

Problem 7 (The Ajima Malfatti point)**.** *The lines joining the point of contact of two Malfatti circles to the vertex opposite the side that is their common tangent, concur at the* **Ajima Malfatti point***.*

Proof. A very direct approach is unlikely to succeed as the vertices of the triangle have little direct role to play in the circles. The **Desargues' Theorem** provides a way out : Let $\triangle ABC$ be the triangle and A', B', C' be the points of tangency of the circles that are opposite A, B, C. We need to show that ABC and $A'B'C'$ are in perspective. In other words we need to determine where $BC \cap B'C'$ and similar points lie and show them to be collinear.

As B' and C' are internal centers of similitude they are collinear with the external center of similitude of the triangles corresponding to B and C (this was one of the variants of Monge's Theorem). Further, as the line BC is itself a direct common tangent it also contains the external center of similitude. Therefore the two lines meet at the external center of similitude.

What thus remains to be seen is that the three external centers of similitude of the circles are collinear, which boils down to an application of **Monge's Theorem**. □

Appendix B. Scope for further exploration

B.1. **Conditional co-incidence problems.** These are problems where we need to prove that the co-incidence of certain varieties is equivalent to the co-incidence of certain other varieties. We have already seen many such results :

- **Desargues' Theorem** states that the concurrence of three lines is equivalent to the collinearity of three points.
- **Simson's Theorem** states that the concyclicity of four points is equivalent to the collinearity of three points.
- Some problems we shall see in the next section, including those on isotomic conjugates, and isogonal conjugates, prove results of the same form : these three lines are concurrent iff those three lines are concurrent.

B.2. **Triangle geometry.** As we had observed right at the outset, triangle geometry provides a rich collection of concurrent lines, concurrent and coaxial circles, and collinear points. The problems in the next section shall explore some of these vistas.

Kimberling has a list of over 500 triangle centers, that can be found on the Internet. For general reading on triangle geometry and other aspects of plane geometry, the following site has plenty of information :

`http://mathworld.wolfram.com/topics/TriangleProperties.html`

Kimberling's triangle centers can be seen on :

`http://faculty.evansville.edu/ck6/tcenters/`

B.3. **Coordinate geometry and geometrical constructions.** In coordinate geometry or for geometrical constructions we try to determine the equation of a line, or circle, based on some conditions that it satisfies.

For instance, we may be interested in constructing (using straight edge and compass) a circle which is tangent to two given lines and passes through a given point.

Alternatively, we may be given the equations of two tangent lines and the coordinates of a point on the circle and we may have to determine the equation of the circle.

We make the following observations :

- The presence of two *independent* data uniquely specify a line, or, at any rate, specify the line upto a finite number of possibilities. For instance, the condition of passing through a given point and being parallel to a given line constitute two independent data that give the line uniquely.

- The presence of three *independent* data uniquely specify a circle, or, at any rate, specify the circle upto a finite number of possibilities. For instance, the condition of passing through three given points fixes the circle uniquely. On the other hand, the condition of being tangent to three given (non-concurrent) lines does not specify a circle uniquely – it gives four possible circles in general (for instance, if no two are parallel, it gives the incircle and the three excircles).

Thus, the condition of being incident on another variety (passing through a given point, or being tangent to a given curve) can be thought of as a single condition (coordinate geometry wise, it gives one equation). This suggests that the incidence number is just a measure of the *number of conditions*.

In coordinate geometry, another way of looking at the number of conditions is as the *number of free parameters* in the general equation of a variety of that type. The general equation of a circle has three free parameters, and the general equation of a line has two free parameters. When we use a different general equation, the nature of the parameters, but the number of free parameters does not change.

Appendix C. Problems

C.1. **Problems based on ideas covered in the article.** Unlike the concept testers, these exercises are *not* made of riders that can be solved immediately based on the material in the text. The exercises are not completely straightforward. However, most of the ideas needed for solving these problems have already been discussed in the text.

(1) **Six point trick** : Let $\triangle ABC$ be a triangle. Let P, Q be points on side BC, R, S on side CA and T, U on side AB. P,Q,R and S are concyclic, R, S, T, and U are concyclic, and T,U, P, and Q are concyclic. **Prove** that P,Q,R,S,T,U are all **concyclic**.

(2) **The contact triangle** : Let $\triangle ABC$ be a triangle and $\triangle DEF$ be its contact triangle (the triangle whose vertices are the points of contact of the incircle with the sides). By convention D is the vertex on BC, E on CA and F on AB. Let P be a point such that AP, BP, CP meet the incircle at U, V, W respectively. Then **show** that DU, EV, FW are **concurrent**. Call the point of concurrence Q. (**Note :** The trilinear coordinates of P and the new point of concurrence are related in an interesting manner)**Source : K.N.Ranganathan**

(3) In the previous exercise determine :
   (a) when P and Q are the same point.
   (b) what triangle center Q is to $\triangle DEF$ when P is the incenter of $\triangle ABC$.

(4) Let A, B, C, D be four distinct points on a line, in that order. The circles with diameters AC and BD meet in X and Y. The line XY meets BC in Z. Let P be a point on XY other than Z. The line CP intersects the circle with diameter AC at C and M and the line BP intersects the circle with diameter BD at B and N. **Prove** that AM, DN, XY are **concurrent**. (*Medium!*) (**Note :** : make diagrams for both the inside and outside cases). **Source : IMO 1995, Problem 1**

(5) Among A, B, C, D no three are collinear. AB and CD meet at E, BC and DA meet at F. **Prove** that either the circles with diameters AC, BD, EF pass through a common point, or no two of them intersect. (*Easy!*) **Source : Hungarian Mathematical Olympiad**

(6) Let α be an acute angle, and $\triangle ABC$ be a triangle. Let $\triangle A'BC$, $\triangle AB'C$, $\triangle ABC'$ be isosceles triangles erected on the sides with base angle α. **Prove** that

AA', BB', CC' **concur**. (**Note :** The locus of this curve as α varies is termed **Kiepert hyperbola** and includes the centroid, the orthocenter, the Napoleon point and the Fermat point. It is the isogonal conjugate of the line joining the isodynamic points).

(7) Let P be a point in $\triangle ABC$ with corresponding Cevians AD, BE, CF. Let Q be a point in $\triangle DEF$ with corresponding Cevians DP, EQ, FR. Show that AP, BQ, CR concur. (*Medium!*) **Source : Challenges and Thrills of Pre College Mathematics**

(8) Given a sequence of points P_i where $1 \leq i \leq n$ and $P_{n+1} = P_1$, **show** that if L is a line and r_i is the signed ratio in which L divides P_iP_{i+1} then $\prod r_i = (-1)^n$. This is the **generalized Menelaus' Theorem** and its converse is not true. (*Easy!*)

(9) Let $\triangle ABC$ be a triangle and A', B', C' be points on BC, CA, AB respectively. Denote by M the point of intersection of circles ABA' and $A'B'C'$ other than A', and by N the point of intersection of circles ABB' and $A'B'C'$ other than B'. Similarly one defines points P, Q, R, S respectively. Then **prove** that :
   (a) At least one of the following is true :
      (i) The triples of lines $(AB, A'M, B'N)$, $(BC, B'P, C'Q)$, $(CA, C'R, A'S)$ are **concurrent** at C'', A'', B" respectively.
      (ii) $A'M$ and $B'N$ are **parallel** to AB, or $B'P$ and $C'Q$ are parallel to BC, or $C'R$ and $A'S$ are parallel to CA.
   (b) In the first case, A'', B'', C'' are collinear. (*Medium!*) **Source : Romanian IMO selection test, 1985**

C.2. **Problems using other techniques.** The problems here make use either of slight variations in the techniques discussed in the article, or use completely new methods. In most of them, some hint is given.

(1) Let Γ be a circle. Let A, B, C and D be points such that AB, BC, CD and DA are all tangent to Γ. **Prove** that the center of Γ and the midpoints of AC and BD are **collinear**. (*Medium!*) (*Hint : Find* the line by treating it as the locus of a point such that certain triangles have equal areas). **Source : I.F.Sharygin**

(2) Let Γ_i be a system of concurrent (not coaxial) circles for $1 \leq i \leq 4$, with P the common point. Let Q_{ij} denote the second intersection point of Γ_i and Γ_j. Let Δ_k be the triangle formed by all the Q_{ij} where $i \neq k \neq j$. Then the circumcircles of Δ_k are again concurrent. (*Medium!*) (*Hint :* Invert about the concurrence point) (**Source : Clifford's Theorem**).

(3) Let $\triangle ABC$ be a triangle. If Γ is an ellipse touching the sides internally (that is, an **inellipse**) – BC at D, CA at E and AB at F, **show** that AD, BE, and CF **concur**. The point of concurrence is termed the **Brianchon point** of the inellipse. (*Medium!*) (*Hint :* Project the triangle to one where the ellipse becomes a circle).

(4) In the previous problem :
   (a) The **Euler inellipse** is the inellipse with foci the orthocenter and circumcenter respectively, and with auxiliary circle the nine point circle. Prove that its Brianchon point is the **isotomic conjugate** of the circumcenter.
   (b) Determine the center of the inellipse with the property that its center is the same as its Brianchon point. This is called the **Steiner inellipse**.

C.3. **Three dimensional variants.** These problems are three dimensional variants of problem directly discussed in the text. The proof techniques used in the text are almost directly applicable in the three dimensional case. The visualization of these problems is somewhat tricky and interesting.

(1) The **altitudes** of a tetrahedron are the perpendiculars from a vertex to the opposite face. An **orthogonal tetrahedron** is one where the altitudes concur. Prove that a tetrahedron $ABCD$ is orthogonal iff $AB^2 + CD^2 = AC^2 + BD^2 = AD^2 + BC^2$.
(2) Define the **radical plane** of two spheres as the locus of all points with equal powers with respect to the two spheres. Show that given three spheres, the radical planes are co-ideal, and intersect in a line. Show further that given four spheres, the four lines obtained thus by taking three spheres at a time are concurrent.
(3) Define the **Appollonius sphere** of two points A and B for a ratio λ as the locus of the point P such that $AP/PB = \lambda$. Show that given three points in space, the Appollonius spheres of each pair containing the third point, meet in a circle.

Appendix D. Definitions

(1) **Tangent** to a curve at a point is a line that most closely approximates the curve at that point. it can be visualized as the straight line path that a particle moving very fast along that curve might take if suddenly freed. This point is termed the **point of contact** of the tangent to the curve. For **convex curves** such as the circle, the tangent does not intersect the curve at any other point. This, however, is not the *general definition* of tangent.
(2) **Common tangent** to two curves is a line that is a tangent to both of them. When both the curves lie on the same side of the line, it is said to be a **direct common tangent** and when both the curves are on opposite sides of the line, it is said to be an **indirect common tangent** or **transverse common tangent**.
(3) **Center of similitude** of two circles is the point that divides the line joining the centers in the ratio of the radii.

    The **internal center of similitude** divides the line internally in the ratio of the radii and is always defined. If the two circles are disjoint, then it is the point of intersection of the indirect common tangents. If the two circles touch externally, it is the common point to them. If the two circles overlap, it lies somewhere in the region of overlap.

    The **external center of similitude** divides the line externally in the ratio of the radii. It is defined when the two circles do not have equal radii. If neither circle is completely inside the other, then it is the point of intersection of the direct common tangents. If the two circles touch internally, then it is the common point to them.
(4) **Envelope** of a collection of lines is a curve to which all the lines are tangents.
(5) **Power of a point** with respect to a circle is given by $d^2 - r^2$ where d is its distance from the center and r is the radius of the circle. It equals the square of the length of the tangent segment from the point to the circle when it lies outside the circle, and is 0 in the circle. It also equals the signed product of its distances from the intersection points of any secant through it, with the circle.

    In coordinate geometry, the power of a point with respect to a circle is obtained by substituting its coordinates in the equation of the circle.
(6) **Radical axis** of two circles is the line comprising all the points having equal powers with respect to the two circles. If the two circles touch, it is the common tangent through the common point. If the two circles meet at two points, it is the **common chord** of the two circles. Otherwise, it intersects neither of the circles.

(7) **Coaxial circles** are circles such that *any two* of them have the same line as their radical axis. A coaxial system is a complete collection of coaxial circles – one to which no other circles can be added.

(8) **Appollonius circle** with respect to two points A and B and corresponding to a ratio λ is the locus of all the points P such that $AP/PB = \lambda$. The Appollonius circle is a straight line (viz the perpendicular bisector of AB) iff $\lambda = 1$. The Appollonius circle has as endpoints of its diameters the points that divide AB internally and externally in the ratio λ.

(9) **Isodynamic points** of a triangle $\triangle ABC$ are the points P satisfying $PA.BC = PB.CA = PC.AB$ and they can also be defined as the common points to three Appollonius circles. They are inverse points with respect to the circumcircle.

(10) **Complete quadrilateral** is a collection of 4 distinct lines, no two of which are parallel and no three of which are concurrent. There is no cyclic ordering implicit in the four lines. Thus, a complete quadrilateral has 6 vertices obtained by taking pairwise intersections of the sides. More on this is found in section 4.5.1.

(11) **Complete quadrangle** is a collection of 4 distinct points, no three of which are collinear. There is no cyclic ordering implicit in the four points. Thus, a complete quadrangle has 6 lines obtained by taking the vertices two at a time. More on this is found in section 4.5.2.

(12) **Parabola** is the locus of a point whose distance from a fixed point, termed the **focus**, equals its perpendicular distance from a fixed line, known as the **directrix**. The **vertex** of the parabola is the midpoint between the focus and its projection on the directrix.

Appendix E. General references

Books (containing relevant theory or Olympiad problems) :

- **Challenges and Thrills of Pre College Mathematics** by V. Krishnamurthy, C.R. Pranesachar, B.J. Venkatachala and K.N. Ranganathan. The geometry section contains extensive coverage of **Ceva's** and **Menelaus' Theorem**.
- **Mathematical Olympiad Challenges** by Titu Andreescu and Razvan Gelca, published by Birkhauser. The geometry section has one part devoted to **cyclic quadrilaterals** and another devoted to **power of a point**. Some of the problems in this article first came to my attention on reading this book.
- **Problems in Plane Geometry** by I.F. Sharygin has problems on all aspects of plane geometry.
- **Problem Solving Strategies** by Arthur Engel, published by Springer. This does not contain any exposition on how to solve geometry problems, but has a number of interesting problems in its plane geometry section.
- **Geometry Revisited** by Coxeter and Greitzer. Chapter 3 is titled "Collinearity and Concurrence".
- **Penguin's Dictionary of Curious and Interesting Geometry** by David Wells.
- Durrell
- Roger Johnson

Websites for reference include :

`http://mathworld.wolfram.com/Geometry.html`

Index