Decision Tree Complexity, Solvable Groups, and the Distribution of Prime Numbers

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Decision Tree Complexity

Decision Tree Model aka Query Model

boolean function $f : \{0,1\}^N \to \{0,1\}$

- <u>Input:</u> $x = (x_1, \ldots, x_N) \in \{0,1\}^N$
- <u>Access:</u> (adaptive) queries $x_i \stackrel{?}{=} 1$
- <u>Cost:</u> #queries
- <u>Goal</u>: determine $f(x)$

<u>Decision Tree Complexity</u>

D(f) := #queries on **worst** input

Evasiveness

$f \in \{0,1\}^N \to \{0,1\}$ is <u>evasive</u>, if

$$D(f) = N$$

A sequence P_n of boolean functions is said to be <u>eventually</u> evasive if P_n is evasive for all sufficiently large n

Note: n and N may be **different**

Graph Properties

Graph Properties as Boolean Functions

Graph Property P_n: collection of n-vertex labeled graphs invariant under relabeling

$$N = \binom{n}{2}$$

n-vertex graph $\longleftrightarrow$ string in $\{0,1\}^N$
graph property $\longleftrightarrow$ $f : \{0,1\}^N \to \{0,1\}$
invariant under relabeling of $[n]$

Examples: planarity, 3-colorability,
complete graphs, connectedness,
Eulerian graphs, perfect graphs

Trivial Graph Properties: all graphs or no graphs

Monotone Graph Properties

- Monotone (Decreasing) Graph Property: closed under edge deletion.

- Examples: planarity, 3-colorability, {empty graph}, trivial graph properties

- Non-monotone Examples: perfect graphs, Eulerian graphs, cycles

Forbidden Subgraph Property
H - a fixed graph

$Q_n^H :=$ the class of all n-vertex graphs that **do not contain** H as a (not necessarily induced) subgraph.

Q_n^H is monotone (decreasing)

Evasiveness Conjecture

Evasiveness Conjecture
aka Aanderaa-Rosenberg-Karp Conjecture for Monotone Graph Properties

For any n, any non-trivial monotone property P_n of n-vertex graphs must be *evasive*, *i.e.*, $D(P_n) = \binom{n}{2}$.

- Aanderaa - Rosenberg Conjecture 1973:
  $D(P_n) = \Omega(n^2)$.

- (Theorem) Rivest and Vuillemin 1978:
  $D(P_n) \geq n^2/16$

- Kahn, Saks, and Sturtevant 1984:
  (KSS Theorem)
  $D(P_n) = \binom{n}{2}$ when n is a prime power
  (conceptual breakthrough: **topological** approach)

KSS Approach

- monotone property : simplicial complex
- group actions on simplicial complex
- Oliver's Theorem:
  group actions $\rightarrow$ fixed points
- fixed point $\rightarrow$ invariant subgraph

Results via KSS topological approach

- Yao 1988: evasiveness in bipartite graphs with fixed partition

- Triesch 1996: bipartite properties (among all graphs)

- Chakrabarti, Khot, and Shi 2002:
  - more tools for Forbidden Subgraph
  - $D(Q_n^H) = \binom{n}{2} - O(n)$
  - forbidden minor : eventually evasive

Our Results

Our Results

- Conditional Results: We confirm under widely accepted number theoretic hypotheses, the eventual evasiveness of
  (a) every forbidden subgraph property
  (b) any monotone property of sparse graphs
  . sparse: $\leq n^{3/2-\epsilon}$ edges

- Unconditional Results
  (a) forbidden sub: $D(Q_n^H) = \binom{n}{2} - O(1)$
  . improves CKS bound: $\binom{n}{2} - O(n)$
  (b) any monotone property of sparse graphs
  . sparse: $\leq cn \log n$

- Unconditional Corollary: forbidden topological subgraph eventually evasive (generalizes CKS: forbidden minor)

Number Theoretic Dependencies

<u>Chowla's Conjecture 1944:</u>
on smallest Dirichlet Prime

$$(\exists p < d^{1+o(1)})(p \equiv a \mod d)$$

<u>Generalized Riemann Hypothesis 1884:</u>
for Dirichlet L-functions

$$(\exists p < d^{2+o(1)})(p \equiv a \mod d)$$

<u>Vinogradov's Theorem 1937:</u>
odd Goldbach Conjecture

$$k \text{ odd} \Rightarrow k = p_1 + p_2 + p_3$$

<u>Haselgrove's Strengthening 1954:</u>
of Vinogradov's Theorem

$$p_1 \approx p_2 \approx p_3$$

<u>Weil's Character Sum Estimates 1941:</u>
for characters of finite field

pseudorandomness of d^{th} power residues

Our Methods

- use KSS topological approach
- use full power of Oliver's Theorem
- new group actions
  via number theory (results/conjectures)
- invariant graphs analysed
  via Weil's character sum estimates
- forbidden subgraph: use CKS reduction of
  Euler characteristic

Some Preliminaries

Abstract Simplicial Complex.

Let $X = \{x_1, \ldots, x_m\}$.
Let $\Delta \subseteq 2^X$ such that:

$$(f \in \Delta)(f' \subseteq f) \Rightarrow f' \in \Delta.$$

Δ : <u>abstract simplicial complex</u>
$f \in \Delta$: <u>face</u> of Δ.

(Dimension) $\dim(\Delta) := \max\{(|f| - 1) \ : \ f \in \Delta\}$.

(Euler Characteristic)

$$\chi(\Delta) := \sum_{f \in \Delta, f \neq \emptyset} (-1)^{|f|-1}.$$

Monotone Property and Abstract Complex

$$[n] := \{1, 2, \ldots, n\}.$$

$$\binom{[n]}{2} := \{\{i, j\} \ : \ (i \neq j \in [n])\}.$$

$\mathcal{G}_n$ - n-vertex (labeled) graphs.

$$G \in \mathcal{G}_n \Rightarrow E(G) \subseteq \binom{[n]}{2}.$$

P_n - monotone (decreasing) graph property

$$\Delta^{P_n} := \{E(G) \ : \ G \in P_n\}$$

$$\dim(P_n) := \dim(\Delta^{P_n})$$

dim = maximum possible #edges - 1

Oliver's Fixed Point Theorem 1976

<u>Oliver's Condition on Group Γ:</u>

$$(\exists\ \Gamma_2, \Gamma_1)(\Gamma_2 \lhd \Gamma_1 \lhd \Gamma)$$

$\exists$ primes p, q such that

- $|\Gamma/\Gamma_1| = q^\beta$

- Γ_1/Γ_2 is cyclic

- $|\Gamma_2| = p^\alpha$

all such groups are <u>solvable</u>; $p = q$ permitted

<u>Oliver's Theorem 1976:</u>
Δ - contractible, non-empty
Γ - satisfies Oliver's Condition
$\Rightarrow$ Γ action on Δ must have
a non-empty <u>Γ-invariant face</u>

Kahn, Saks, and Sturtevant's Approach

$\boxed{\text{Evasiveness} \leftarrow \text{Topology} + \text{Group Actions}}$

- P_n <u>not evasive</u> $\Rightarrow \Delta^{P_n}$ <u>contractible</u>
  $(\Rightarrow \chi(P_n) = 1)$

- <u>Graph Property:</u> $\Gamma \leq S_n \Rightarrow \Gamma$ acts on Δ^{P_n}.

- use Oliver's Fixed Point Theorem

KSS Theorem

Theorem (KSS 1984): If $n = p^\alpha$, then any non-evasive monotone graph property P_n satisfied by at least one graph is satisfied by K_n, and hence by all graphs.

Key use of prime power: There exists a group $\Gamma \leq S_n$ such that:

(a) Γ satisfies Oliver's Condition

(b) Γ is transitive on $\binom{[n]}{2}$

Γ acts on the non-empty contractible simplicial complex Δ^{P_n} and via Oliver, Δ^{P_n} has a non-empty Γ-invariant face

Since Γ is transitive on $\binom{[n]}{2}$, this Γ-invariant face must be K_n. Thus, $K_n \in \Delta^{P_n}$

KSS Group Construction for $n = p^\alpha$
$\Gamma := \mathbb{F}_{p^\alpha}^{+} \rtimes \mathbb{F}_{p^\alpha}^{\times}$: affine group over $\mathbb{F}_{p^\alpha}$
Explicitly

- identify $[n]$ with $\mathbb{F}_{p^\alpha}$
- $\Gamma := \{\gamma : x \mapsto ax + b \mid a \in \mathbb{F}_{p^\alpha}^{\times} \ \& \ b \in \mathbb{F}_{p^\alpha}^{+}\}$
- Γ - cyclic extension of a p-group
  (thus Γ satisfies Oliver's Condition)
- Γ - transitive on $\binom{[n]}{2}$

Our Approach to Sparse Graphs

Evasiveness and Dimension

A Restatement of Evasiveness Conjecture:

$(P_n \neq \emptyset)(P_n$ not evasive$) \Rightarrow \mathrm{dim}(P_n) = \binom{n}{2} - 1.$

Our Sparse Graph Results:
$(P_n \neq \emptyset)(P_n$ not evasive$) \Rightarrow \mathrm{dim}(P_n) \geq m(n)$

in other words ...
any non-trivial monotone property of graphs with at most $m(n)$ edges is evasive
stronger number theory $\Rightarrow$ larger $m(n)$

Dimension Lower Bound

trivial bound via KSS : $\Omega(n)$

We show

- $\Omega(n \log n)$ <u>unconditionally</u>

- $\Omega(n^{5/4-\epsilon})$ <u>under GRH</u>

- $\Omega(n^{3/2-\epsilon})$ <u>under Chowla's Conjecture</u>

still far from quadratic

This translates to ...

property of graphs with at most m edges $\equiv$ property fails for any graph having $> m$ edges

any non-trivial monotone property of graphs with at most m edges is eventually evasive where

- $m = cn \log n$ unconditionally
- $m = n^{5/4-\epsilon}$ under GRH
- $m = n^{3/2-\epsilon}$ under Chowla's Conjecture

Our Approach to Sparse Graphs:

Construct $\Gamma \leq S_n$ such that:

- Γ satisfies <u>Oliver's Condition</u>

- size of the <u>orbit of any edge</u> under Γ is as <u>large</u> as possible

<u>KSS 84</u>

$\boxed{\text{Evasiveness} \leftarrow \text{Topology} + \boxed{\text{Group Actions}}}$

<u>Our new component</u>

$\boxed{\boxed{\text{Group Actions}} \leftarrow \text{Analytic Number Theory}}$

Forbidden Subrgraph

CKS Approach to Forbidden Subgraph
Chakrabarti, Khot, and Shi 2002:

- use KSS + Oliver

- construct new group actions
  specific to Forbidden Subgraph

- invariant graph trivially contains
  a large clique

Our New Components to CKS Approach

- metabelian group actions
  to force large clique non-trivially

- Paley graphs

- Weil's character sum estimates

- use distribution of prime numbers
  (known/conjectured) to glue the pieces

Paley-type Graphs & Metabelian Groups
Construction of Graph $P(q, d)$

- $V = \mathbb{F}_q$ $\quad$ q odd prime power
  d even $\quad$ $d \mid q - 1$

- $i \sim j \iff (i - j)^d = 1$ (over $\mathbb{F}_q$)

$\Gamma(q, d) :=$ order qd subgroup of $\mathbb{F}_q^+ \rtimes \mathbb{F}_q^\times$

Main Observations

- orbit of any (unordered) pair $\{i, j\} \in \binom{[q]}{2}$ under $\Gamma(q, d)$ action is isomorphic to $P(q, d)$

- if $\frac{q-1}{d} \leq q^{1/2h}$ then $P(q, d)$ contains a clique on h vertices

Paley-type graphs **pseudorandom**

proof goes via standard application of Weil's Character Sum Estimates

More Details : Sparse Graphs

(mostly skipping ...)

A Prime-Partition of k.

Goldbach Conjecture:
k - even integer $\Rightarrow k = p_1 + p_2$
p_i prime.

Vinogradov's **Theorem**:
k - large odd integers
$\Rightarrow k = p_1 + p_2 + p_3$

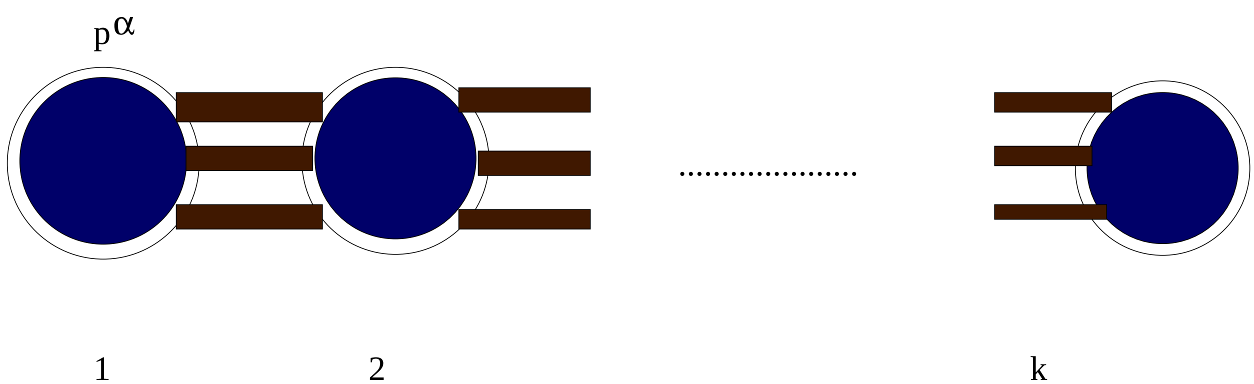
Haselgrove's Strengthening: $\boxed{p_i = \Omega(k)}$

Corollary: k large even integer
$\Rightarrow k = p_1 + p_2 + p_3 + p_4$,
$\boxed{p_i = \Omega(k)}$

Partition of $\{1, 2, \ldots, n\}$

Let $n = p^{\alpha}k \qquad p$ prime

$|S| = n.$



$$[n] = \underbrace{\mathbb{F}_{p^{\alpha}} \dot{\cup} \ldots \dot{\cup} \mathbb{F}_{p^{\alpha}}}_{k}$$

Our Basic Group Construction

Γ acts on $[n]$ such that

- within each block Γ simulates action of affine group over $\mathbb{F}_{p^\alpha}$: $x \mapsto ax + b$

- $k = p_1 + p_2 + p_3$ where $p_i \approx p_j$ (Vinogradov + Haselgrove)

$$H := \mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2} \times \mathbb{Z}_{p_3}$$

- Γ permutes blocks according to $H \leq S_k$

$$\overbrace{\underbrace{00\ldots0}_{p_1}\underbrace{00\ldots0}_{p_2}\underbrace{00\ldots0}_{p_3}}^{k}$$

- Γ satisfies Oliver's Condition (not obvious)

Description of Γ

$$n = p^\alpha k \qquad H \leq S_k$$

$$\Gamma := (\underbrace{\mathbb{F}_{p^\alpha}^{+} \times \cdots \times \mathbb{F}_{p^\alpha}^{+}}_{k}) \rtimes (\mathbb{F}_{p^\alpha}^{\times} \times H)$$

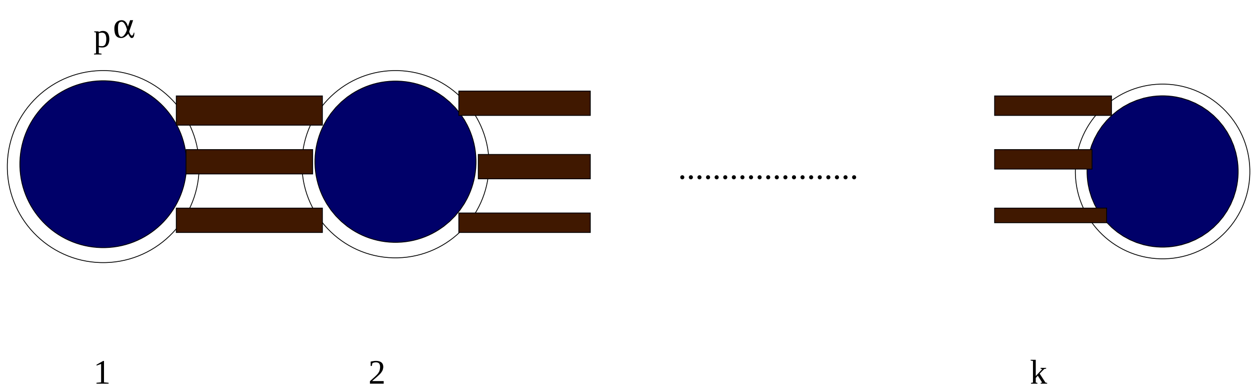
Orbit of an Edge

Two types of edges

- <u>intra-cluster</u> : orbit K_{p^α}

- <u>inter-cluster</u> : orbit K_{p^α, p^α}

intra-cluster edge orbits $\leftrightarrow$ H-orbits on $[k]$ inter-cluster edge orbits $\leftrightarrow$ H-orbits on $\binom{[k]}{2}$



Orbit Size Lower Bounds

$k = p_1 + p_2 + p_3 \quad p_i \approx p_j \quad p_1 \leq p_2 \leq p_3$

- intra-cluster edge -

$$|\text{intra-cluster orbit}| \geq \binom{p^\alpha}{2} \times p_1$$

- inter-cluster edge -

$$|\text{inter-cluster orbit}| \geq (p^\alpha)^2 \times p_1$$

- any edge -

$$|\text{any orbit}| = \Omega(p^\alpha \times p^\alpha \times p_1)$$

Choice of p^α

$$p^\alpha := \text{largest prime power dividing } n$$

- $p^\alpha = \Omega(\log n)$
- $n \sim 3p^\alpha p_1$
- $|\text{any orbit}| = \Omega(n \log n)$

(thanks to Vinogradov's **Theorem**)
this proves **unconditionally** ...

there exists a constant c such that any monotone property of graphs with $\leq cn \log n$ edges is evasive

Another Partition of n

We want to write

$$n = pk + r$$

such that

$$p,\ r\ prime.$$

$$p = \Theta(n^{1/4})$$

$$\frac{n}{4} \le r \le \frac{n}{2}$$

$$(\exists q)(q\ prime)(q \mid r - 1)(q = \Theta(n^{1/4-\epsilon}))$$

GRH and Dirichlet Primes

For a fixed D and a such that gcd$(a, D) = 1$ there are infinitely many primes of the form $p \equiv a$ mod D.

Under GRH:
If $D = O(n^{1/2-\epsilon})$, then
for any a such that $gcd(a, D) = 1$,
there exists a prime $p \equiv a$ mod D
such that $\frac{n}{2} \leq p \leq n$.

GRH $\Rightarrow$ desired partition

Choose some prime $p = \Theta(n^{1/4})$.
Choose another prime $q = \Theta(n^{1/4-\epsilon})$.
We need to find a prime r such that

$$r \equiv n \bmod p \ \& \ r \equiv 1 \bmod q$$

Equivalently,
for some a such that $gcd(a, pq) = 1$
we want to find $\frac{n}{4} \leq r \leq \frac{n}{2}$ such that

$$r \equiv a \bmod pq.$$

Since $pq = O(n^{1/2-\epsilon})$,
GRH $\Rightarrow$ such r exists.

Choosing Γ

$$\Gamma := \Gamma_{[pk]} \times \Gamma_r$$

where

$$\Gamma_r := \mathbb{F}_r^+ \rtimes \mathbb{Z}_q$$

$\Gamma_{[pk]}$ - as constructed previously using prime partition of k.

With this delicate choice of parameters, Γ satisfies Oliver's Condition and one can show:

$(P_n \neq \emptyset)(P_n$ not evasive$) \Rightarrow \dim(P_n) = \Omega(n^{5/4-\epsilon})$

Possible Directions

- first try to resolve the following:
  (under number theoretic conjectures)

  $(P_n \neq \emptyset)(P_n \text{ not evasive}) \Rightarrow \dim(P_n) = \Omega(n^2)$

- prove evasivenes conjecture or
  strong dimension lower bounds
  on sets of positive density

- unconditional result for Forbidden Subgraph

Thanks !