TAKE-HOME CLASS QUIZ: DUE MONDAY DECEMBER 2: SIMILARITY OF LINEAR TRANSFORMATIONS (APPLIED)

MATH 196, SECTION 57 (VIPUL NAIK)

Your name (print clearly in capital letters): ____________________

PLEASE FEEL FREE TO DISCUSS *ALL* QUESTIONS.

This quiz corresponds to material discussed in the lecture notes titled `Coordinates`. It also corresponds to Section 3.4 of the text.

Recall that $n \times n$ matrices A and B are termed *similar* if there exists an invertible $n \times n$ matrix S such that $A = SBS^{-1}$. The relation of matrices being similar is an *equivalence relation*.

Recall that $n \times n$ matrices A and B are termed *quasi-similar* if there exist $n \times n$ matrices C and D such that $A = CD$ and $B = DC$. Recall that similar matrices are always quasi-similar, but quasi-similar matrices need not be similar. However, for *invertible* matrices, similarity and quasi-similarity are equivalent.

Also, note that if A and B are quasi-similar matrices, then A and B have the same trace. However, the converse is not true: it is possible to have two matrices A and B that have the same trace but are not quasi-similar.

For these questions, assume $n > 1$, because a lot of phenomena are much simpler in the case $n = 1$ and you may be misled if you look only at that case.

Note also that the trace of a square matrix is defined as the sum of its diagonal entries.

The *determinant* of a 2×2 matrix, denoted det, is defined as follows:

$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$$

The following are some important facts about the determinant:

- The determinant of a 2×2 diagonal matrix is the product of the diagonal entries.
- The determinant of a 2×2 matrix is zero if and only if the matrix is non-invertible.
- The determinant of the product of two 2×2 matrices is the product of the determinants.
- The determinant of the inverse of an invertible 2×2 matrix is the reciprocal of the determinant.
- If A and B are similar 2×2 matrices, they have the same determinant.
- If A and B are quasi-similar 2×2 matrices, they have the same determinant.
- If the determinant of A is positive, then the linear transformation given by A is an orientation-preserving linear automorphism of $\mathbb{R}^2$.
- If the determinant of A is negative, then the linear transformation given by A is an orientation-reversing linear automorphism of $\mathbb{R}^2$.

(1) Suppose A and B are both $n \times n$ matrices (but they are not given to be similar). Denote by I_n the $n \times n$ identity matrix. Which of the following holds?
  (A) A is similar to B if and only if $A - I_n$ is similar to $B - I_n$.
  (B) If A is similar to B, then $A - I_n$ is similar to $B - I_n$. However, $A - I_n$ being similar to $B - I_n$ does not imply that A is similar to B.
  (C) If $A - I_n$ is similar to $B - I_n$, then A is similar to B. However, A being similar to B does not imply that $A - I_n$ is similar to $B - I_n$.
  (D) A being similar to B does not imply that $A - I_n$ is similar to $B - I_n$. Also, $A - I_n$ being similar to $B - I_n$ does not imply that A is similar to B.

  Your answer: ____________________

Suppose f is a polynomial of degree r in one variable with real coefficients. For a $n \times n$ matrix X, we denote by $f(X)$ we mean the matrix we get by applying the polynomial to f, where constant terms are interpreted as scalar matrices. For instance, if $f(x) = x^2 + 3x + 5$, then $f(X) = X^2 + 3X + 5I_n$.

(2) Suppose A and B are both $n \times n$ matrices (but they are not given to be similar). Suppose f is a polynomial of degree r in one variable, where $r \geq 2$. Which of the following holds?
(A) A is similar to B if and only if $f(A)$ is similar to $f(B)$.
(B) If A is similar to B, then $f(A)$ is similar to $f(B)$. However, $f(A)$ being similar to $f(B)$ does not imply that A is similar to B.
(C) If $f(A)$ is similar to $f(B)$, then A is similar to B. However, A being similar to B does not imply that $f(A)$ is similar to $f(B)$.
(D) A being similar to B does not imply that $f(A)$ is similar to $f(B)$. Also, $f(A)$ being similar to $f(B)$ does not imply that A is similar to B.

Your answer: ____________________________

(3) Suppose A and B are both $n \times n$ matrices (but they are not given to be similar). Suppose f is a polynomial of degree r in one variable, where $r = 1$. Which of the following holds?
(A) A is similar to B if and only if $f(A)$ is similar to $f(B)$.
(B) If A is similar to B, then $f(A)$ is similar to $f(B)$. However, $f(A)$ being similar to $f(B)$ does not imply that A is similar to B.
(C) If $f(A)$ is similar to $f(B)$, then A is similar to B. However, A being similar to B does not imply that $f(A)$ is similar to $f(B)$.
(D) A being similar to B does not imply that $f(A)$ is similar to $f(B)$. Also, $f(A)$ being similar to $f(B)$ does not imply that A is similar to B.

Your answer: ____________________________

Suppose p and q are real numbers (possibly equal, possibly distinct). The diagonal matrices:

$$A = \begin{bmatrix} p & 0 \\ 0 & q \end{bmatrix}$$

and

$$B = \begin{bmatrix} q & 0 \\ 0 & p \end{bmatrix}$$

are similar. Explicitly, the two matrices are similar under the change-of-basis transformation that interchanges the coordinates, i.e., if we set:

$$S = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

then:

$$S = S^{-1}$$

and we have:

$$B = S^{-1}AS$$

Moreover, the only diagonal matrices similar to A are A and B (in the special case that $p = q$, we get $A = B$ is a scalar matrix, so A is the only diagonal matrix similar to A).

(4) What is the necessary and sufficient condition on p and q such that the matrix $A = \begin{bmatrix} p & 0 \\ 0 & q \end{bmatrix}$ is similar to $-A$?

(A) $p = q$
(B) $p = -q$
(C) $p = 1/q$
(D) $p = -1/q$
(E) $p + q = 1$

Your answer: ____________________________

(5) Which of the following is a necessary and sufficient condition on p and q so that the matrix $A = \begin{bmatrix} p & 0 \\ 0 & q \end{bmatrix}$ is invertible and similar to $-A^{-1}$?
(A) $p = q$
(B) $p = -q$
(C) $p = 1/q$
(D) $p = -1/q$
(E) $p + q = 1$

Your answer: ____________________________

Consider the matrix:

$$S = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

used above. We have $S = S^{-1}$. For a general matrix:

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

we have:

$$S^{-1}AS = \begin{bmatrix} d & c \\ b & a \end{bmatrix}$$

In other words, it swaps the rows *and* swaps the columns. This observation may be useful for some of the following questions.

(6) For an angle θ with $-\pi \leq \theta \leq \pi$, the rotation matrix for θ is given as:

$$R(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

Note that $R(-\pi) = R(\pi)$, but other than that equality, all the $R(\theta)$s are distinct.

Which of these describes the relation between the rotation matrices for different values of θ?
(A) All the rotation matrices $R(\theta)$, $-\pi < \theta \leq \pi$, are similar to each other.
(B) The rotation matrix $R(\theta)$ is similar to itself and to the rotation matrix $R(-\theta)$. However, it is not in general similar to any other rotation matrix.
(C) No two different rotation matrices are similar.
(D) The rotation matrix $R(\theta)$ is similar to itself and to the rotation matrix $R(\pi - \theta)$ (or $R(-\pi - \theta)$, depending on which of the two angles lies within the specified range). However, it is not in general similar to any other rotation matrix.

Your answer: ____________________________

(7) Consider the linear automorphisms of $\mathbb{R}^2$ that are given as *reflections* about lines in $\mathbb{R}^2$ through the origin. (Note that we need the line of reflection to pass through the origin for the automorphism to be *linear* rather than merely being *affine linear*). Which of these describes the relation between reflection matrices for different possible lines of reflection through the origin?

(A) All the reflection matrices are similar to each other.
(B) No two reflection matrices for different lines of reflection are similar.
(C) The reflection matrices for two different lines of reflection are similar if and only if the lines of reflection are perpendicular.
(D) The reflection matrices for two different lines of reflection are similar if and only if the lines are reflection make an angle that is a rational multiple of π.

Your answer: ____________________________

(8) Suppose m and n are positive integers with $m < n$. Denote by P_m the "orthogonal projection onto the first m coordinates" linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^n$, defined as follows. This takes as input a n-dimensional vector, sends each of the first m coordinates to itself, and sends the remaining coordinates to zero. What is the trace of the matrix of P_m?
(A) 1
(B) m
(C) n
(D) $n - m$
(E) $m - n$

Your answer: ____________________________

(9) It is a fact that if A, B are $n \times n$ matrices that describe orthogonal projections onto (possibly different) m-dimensional subspaces of $\mathbb{R}^n$, then A and B are similar. What can we say must be the trace of an orthogonal projection onto any m-dimensional subspace of $\mathbb{R}^n$?
(A) 1
(B) m
(C) n
(D) $n - m$
(E) $m - n$

Your answer: ____________________________

(10) Suppose A, B and C are $n \times n$ matrices. Which of the following matrices is *not* guaranteed (based on the given information) to have the same trace as the product ABC? Please see (and read carefully) Options (D) and (E) before answering.
(A) BCA
(B) CBA
(C) CAB
(D) None the above, i.e., they are all guaranteed to have the same trace as ABC.
(E) All of the above, i.e., none of them is guaranteed to have the same trace as ABC.

Your answer: ____________________________

(11) Which of the following gives a pair of matrices A and B that have the same trace as each other *and* the same determinant as each other, but that are *not* similar to each other?
(A) $A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, B = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$
(B) $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$
(C) $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$
(D) $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$
(E) $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, B = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$

Your answer: ____________________________

(12) Suppose A and B are 2×2 matrices. Which of the following correctly describes the relation between $\det A$, $\det B$, and $\det(A+B)$? Please see Option (E) before answering.
(A) $\det(A+B) = \det A + \det B$
(B) $\det(A+B) \le \det A + \det B$, but equality need not necessarily hold.
(C) $\det(A+B) \ge \det A + \det B$, but equality need not necessarily hold.
(D) $|\det(A+B)| \le |\det A| + |\det B|$, but equality need not necessarily hold.
(E) None of the above.

Your answer: ____________________________

Let n be a natural number greater than 1. Suppose $f : \{0,1,2,\dots,n\} \to \{0,1,2,\dots,n\}$ is a function satisfying $f(0)=0$. Let T_f denote the linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^n$ satisfying the following for all $i \in \{1,2,\dots,n\}$:

$$T_f(\vec{e}_i) = \begin{cases} \vec{e}_{f(i)}, & f(i) \neq 0 \\ \vec{0}, & f(i) = 0 \end{cases}$$

Let M_f denote the matrix for the linear transformation T_f. M_f can be described explicitly as follows: the i^{th} column of M_f is $\vec{0}$ if $f(i)=0$ and is $\vec{e}_{f(i)}$ if $f(i) \neq 0$.

Note that if $f, g : \{0,1,2,\dots,n\} \to \{0,1,2,\dots,n\}$ are functions with $f(0)=g(0)=0$, then $M_{f\circ g} = M_f M_g$ and $T_{f\circ g} = T_f \circ T_g$.

For the following questions, the discussion prior to Question 3 might be helpful. Note, however, that while that discussion gives one possible candidate for the matrix S of the similarity transformation, it is not the only possible candidate. For some but not all of the following questions, in the case that two matrices are similar, the matrix S described there works. In the case that they are not similar, the lack of similarity can be inferred from the traces not being equal, or from the determinants not being equal.

(13) $n=2$ for this question. For the following three functions f, g, and h, consider the corresponding matrices M_f, M_g, M_h. Either two of them are similar and the third is not similar to either (in which case, select the matrix that is not similar to the other two), or all three are similar (if so, select Option (D)), or no two are similar (if so, select Option (E)).
(A) $f(0)=0, f(1)=1, f(2)=0$
(B) $g(0)=0, g(1)=0, g(2)=2$
(C) $h(0)=0, h(1)=1, h(2)=1$
(D) All the above give similar matrices.
(E) No two of the corresponding matrices are similar.

Your answer: ____________________________

(14) $n=2$ for this question. For the following three functions f, g, and h, consider the corresponding matrices M_f, M_g, M_h. Either two of them are similar and the third is not similar to either (in which case, select the matrix that is not similar to the other two), or all three are similar (if so, select Option (D)), or no two are similar (if so, select Option (E)).
(A) $f(0)=0, f(1)=0, f(2)=1$
(B) $g(0)=0, g(1)=2, g(2)=0$
(C) $h(0)=0, h(1)=2, h(2)=1$
(D) All the above give similar matrices.
(E) No two of the corresponding matrices are similar.

Your answer: ____________________________

(15) $n=3$ for this question. For the following three functions f, g, and h, consider the corresponding matrices M_f, M_g, M_h. Either two of them are similar and the third is not similar to either (in which case, select the matrix that is not similar to the other two), or all three are similar (if so, select Option (D)), or no two are similar (if so, select Option (E)).

(A) $f(0) = 0, f(1) = 2, f(2) = 1, f(3) = 3$
(B) $g(0) = 0, g(1) = 1, g(2) = 3, g(3) = 2$
(C) $h(0) = 0, h(1) = 3, h(2) = 2, h(3) = 1$
(D) All the above give similar matrices.
(E) No two of the corresponding matrices are similar.

Your answer: ____________________________

(16) $n = 3$ for this question. For the following three functions f, g, and h, consider the corresponding matrices M_f, M_g, M_h. Either two of them are similar and the third is not similar to either (in which case, select the matrix that is not similar to the other two), or all three are similar (if so, select Option (D)), or no two are similar (if so, select Option (E)).
(A) $f(0) = 0, f(1) = 1, f(2) = 2, f(3) = 3$
(B) $g(0) = 0, g(1) = 2, g(2) = 3, g(3) = 1$
(C) $h(0) = 0, h(1) = 3, h(2) = 1, h(3) = 2$
(D) All the above give similar matrices.
(E) No two of the corresponding matrices are similar.

Your answer: ____________________________