TAKE-HOME CLASS QUIZ: DUE MONDAY NOVEMBER 25: SUBSPACE, BASIS, DIMENSION, AND ABSTRACT SPACES: APPLICATIONS TO CALCULUS

MATH 196, SECTION 57 (VIPUL NAIK)

Your name (print clearly in capital letters): ______________________________________

PLEASE FEEL FREE TO DISCUSS *ALL* QUESTIONS.

This quiz builds on the November 8 and November 20 quizzes that apply ideas we are learning about linear transformations to the calculus setting. The November 8 quiz went over some basic ideas related to differentiation as a linear transformation. The November 20 quiz explored the ideas in greater depth. We now look at questions that apply the ideas of basis, dimension, and subspace to the calculus setting.

We begin by recalling some notation and facts we already saw in earlier quizzes. Denote by $C(\mathbb{R})$ (or alternatively by $C^0(\mathbb{R})$) the vector space of all continuous functions from $\mathbb{R}$ to $\mathbb{R}$, with pointwise addition and scalar multiplication. Note that the elements of this vector space, which we would ordinarily call "vectors", are now *functions*.

For k a positive integer, denote by $C^k(\mathbb{R})$ the subspace of $C(\mathbb{R})$ comprising those continuous functions that are at least k times *continuously* differentiable. Note that $C^{k+1}(\mathbb{R})$ is a subspace of $C^k(\mathbb{R})$, so we have a descending chain of subspaces:

$$C(\mathbb{R}) = C^0(\mathbb{R}) \supseteq C^1(\mathbb{R}) \supseteq C^2(\mathbb{R}) \supseteq \dots$$

The intersection of these spaces is the vector space $C^\infty(\mathbb{R})$, defined as the subspace of $C(\mathbb{R})$ comprising those functions that are *infinitely* differentiable.

We had also noted that:

- The kernel of differentiation is the vector space of constant functions.
- The kernel of k times differentiating is the vector space of polynomials of degree at most $k-1$.
- The fiber of any function for differentiation is a translate of the space of constant functions. That's what explains the $+C$ when you perform indefinite integration.

Note: For finite-dimensional spaces, a linear transformation T from a vector space to itself is injective if and only if it is surjective. This follows from dimension and rank considerations: T is injective if and only its kernel is zero, which happens if and only if the matrix has full column rank, which happens if and only if the matrix has full row rank (because the matrix is a square matrix), which happens if and only if T is surjective. The rank-nullity theorem provides an equivalent explanation. We had also seen that if $T : \mathbb{R}^m \to \mathbb{R}^n$ is injective, then $m \le n$, and if $T : \mathbb{R}^m \to \mathbb{R}^n$ is surjective, then $m \ge n$. In particular, we cannot have a surjective map from a proper subspace to the whole space.

With infinite-dimensional spaces, however, we can have funny phenomena. Examples of these phenomena are strewn across the quizzes.

- We can have a map from an infinite-dimensional vector space to itself that is injective but not surjective.
- We can have a map from an infinite-dimensional vector space to itself that is surjective but not injective.
- We can have a surjective map from a proper subspace to the whole space (for instance, differentiation $C^1(\mathbb{R}) \to C(\mathbb{R})$ is surjective, even though $C^1(\mathbb{R})$ is a proper subspace of $C(\mathbb{R})$).
- We can have an injective map from a space to a proper subspace.

Note that we will use the terms *subspace* and *vector subspace* synonymously with *linear subspace* in this quiz.

(1) Suppose V is a vector subspace of the vector space $C^\infty(\mathbb{R})$. We know that differentiation is linear. How is that information computationally useful?
   (A) It tells us that knowing how to differentiate all functions in any spanning set for V tells us how to differentiate any function in V (assuming we know how to express any function in V as a linear combination of the functions in the spanning set).
   (B) It tells us that knowing how to differentiate all functions in any linearly independent set in V tells us how to differentiate any function in V.

   Your answer: ____________________________

(2) Suppose V is a vector subspace of the vector space $C^\infty(\mathbb{R})$. We know that differentiation is linear. How is that information computationally useful?
   (A) It tells us that knowing the antiderivatives of all functions in any spanning set for V tells us the antiderivative of every function in V (assuming we know how to express any function in V as a linear combination of the functions in the spanning set).
   (B) It tells us that knowing the antiderivatives of all functions in any linearly independent set in V tells us the antiderivative of every function in V.

   Your answer: ____________________________

We now consider two related vector spaces. $\mathbb{R}[x]$ is defined as the vector space of polynomials with real coefficients in the single variable x, with the usual addition and scalar multiplication. There is a natural injective homomorphism from $\mathbb{R}[x]$ to $C^\infty(\mathbb{R})$ that sends any polynomial to the same polynomial viewed as a function.

$\mathbb{R}(x)$ is defined as the vector space of all rational functions where the numerator and denominator are both polynomials with the denominator nonzero, up to equivalence (i.e., two rational functions $p_1(x)/q_1(x)$ and $p_2((x)/q_2(x)$ are equivalent if $p_1(x)q_2(x) = q_1(x)p_2(x)$). Addition and scalar multiplication are defined the usual way. Note that there is a natural injective homomorphism from $\mathbb{R}[x]$ to $\mathbb{R}(x)$ that sends any polynomial $p(x)$ to the rational function $p(x)/1$.

Also note that $\mathbb{R}(x)$ does not map to $C^\infty(\mathbb{R})$, for the reason that a rational function, viewed *qua* function, is not necessarily defined everywhere. Specifically, if written in simplified form, it is not defined at the set of roots of its denominator.

Note that both $\mathbb{R}[x]$ and $\mathbb{R}(x)$ are infinite-dimensional vector spaces, i.e., they do not have finite spanning sets.

(3) Which of the following is *not* a basis for $\mathbb{R}[x]$? Please see Option (E) before answering.
   (A) $1, x, x^2, x^3, \dots$
   (B) $1, x, x(x-1), x(x-1)(x-2), x(x-1)(x-2)(x-3), \dots$
   (C) $1, x+1, x^2+x+1, x^3+x^2+x+1, \dots$
   (D) $1, x, x^2-x, x^3-x^2, x^4-x^3, \dots$
   (E) None of the above, i.e., each of them is a basis.

   Your answer: ____________________________

Let's now revisit the topic of *partial fractions* as a tool for integrating rational functions. The idea behind partial fractions is to consider an integration problem with respect to a variable x with integrand of the following form:

$$\frac{a_0 + a_1x + a_2x^2 + \cdots + a_{n-1}x^{n-1}}{p(x)}$$

where p is a polynomial of degree n. For convenience, we may take p to be a monic polynomial, i.e., a polynomial with leading coefficient 1. For p fixed, the set of all rational functions of the form above forms a vector subspace of dimension n inside $\mathbb{R}(x)$. A natural choice of basis for this subspace is:

$$\frac{1}{p(x)}, \frac{x}{p(x)}, \dots, \frac{x^{n-1}}{p(x)}$$

The goal of partial fraction theory is to provide an *alternate basis* for this space of functions with the property that those basis elements are particularly easy to integrate (recurring to one of our earlier questions). Let's illustrate one special case: the case that p has n distinct real roots $\alpha_1, \alpha_2, \dots, \alpha_n$. The alternate basis in this case is:

$$\frac{1}{x-\alpha_1}, \frac{1}{x-\alpha_2}, \dots, \frac{1}{x-\alpha_n}$$

The explicit goal is to rewrite a partial fraction:

$$\frac{a_0 + a_1 x + a_2 x^2 + \cdots + a_{n-1}x^{n-1}}{p(x)}$$

in terms of the basis above. If we denote the numerator as $r(x)$, we want to write:

$$\frac{r(x)}{p(x)} = \frac{c_1}{x-\alpha_1} + \frac{c_2}{x-\alpha_2} + \cdots + \frac{c_n}{x-\alpha_n}$$

The explicit formula is:

$$c_i = \frac{r(\alpha_i)}{\prod_{j \neq i}(\alpha_i - \alpha_j)}$$

Once we rewrite the original rational function as a linear combination of the new basis vectors, we can integrate it easily because we know the antiderivatives of each of the basis vectors. The antiderivative is thus:

$$\left(\sum_{i=1}^{n} \frac{r(\alpha_i)}{\prod_{j \neq i}(\alpha_i - \alpha_j)} \ln|x - \alpha_i| \right) + C$$

where the obligatory $+C$ is put for the usual reasons.

Note that this process only handles rational functions that are proper fractions, i.e., the degree of the numerator must be less than that of the denominator.

We now consider cases where p is a polynomial of a different type.

(4) Suppose p is a monic polynomial of degree n that is a product of pairwise distinct irreducible factors that are all either monic linear or monic quadratic. Call the roots for the linear polynomials $\alpha_1, \alpha_2, \dots, \alpha_s$ and call the monic quadratic factors $q_1, q_2, \dots, q_t$. Which of the following sets forms a basis for the vector space that we are interested in, namely all rational functions of the form $r(x)/p(x)$ where the degree of r is less than n? Please see Option (E) before answering.
(A) All rational functions of the form $1/(x - \alpha_i), 1 \leq i \leq s$ together with all rational functions of the form $1/q_j(x), 1 \leq j \leq t$
(B) All rational functions of the form $1/(x - \alpha_i), 1 \leq i \leq s$ together with all rational functions of the form $q_j'(x)/q_j(x), 1 \leq j \leq t$
(C) All rational functions of the form $1/q_j(x), 1 \leq j \leq t$ together with all rational functions of the form $q_j'(x)/q_j(x), 1 \leq j \leq t$
(D) All rational functions of the form $1/(x - \alpha_i), 1 \leq i \leq s$ together with all rational functions of the form $1/q_j(x), 1 \leq j \leq t$ *and* all rational functions of the form $q_j'(x)/q_j(x), 1 \leq j \leq t$
(E) None of the above

Your answer: ____________________________

(5) Suppose $p(x) = (x - \alpha)^n$. Which of the following sets forms a basis for the vector space that we are interested in, namely all rational functions of the form $r(x)/p(x)$ where the degree of r is less than n? Please see Options (D) and (E) before answering.
(A) The single function $1/(x - \alpha)$
(B) The single function $1/(x - \alpha)^n$
(C) All the functions $1/(x - \alpha), 1/(x - \alpha)^2, \dots, 1/(x - \alpha)^n$
(D) Any of the above works

(E) None of the above works

Your answer: ____________________________

We now recall our earlier discussion of the solution process for first-order linear differential equations. Consider a first-order linear differential equation with independent variable x and dependent variable y, with the equation having the form:

$$y' + p(x)y = q(x)$$

where $p, q \in C^\infty(\mathbb{R})$.
We solve this equation as follows. Let H be an antiderivative of p, so that $H'(x) = p(x)$.

$$\frac{d}{dx}\left(ye^{H(x)}\right) = q(x)e^{H(x)}$$

This gives:

$$ye^{H(x)} = \int q(x)e^{H(x)}\,dx$$

So:

$$y = e^{-H(x)}\int q(x)e^{H(x)}\,dx$$

The indefinite integration gives a $+C$, so overall, we get:

$$y = Ce^{-H(x)} + \text{particular solution}$$

It's now time to understand this in terms of linear algebra.
Define a linear transformation $L : C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R})$ as:

$$f(x) \mapsto f'(x) + p(x)f(x)$$

(6) The kernel of L is one-dimensional. Which of the following functions spans the kernel?
(A) $p(x)$
(B) $q(x)$
(C) $H(x)$
(D) $e^{H(x)}$
(E) $e^{-H(x)}$

Your answer: ____________________________

(7) I would like to argue that L is *surjective* as a linear transformation from $C^\infty(\mathbb{R})$ to $C^\infty(\mathbb{R})$. Why is that true?
(A) The kernel of L is zero-dimensional.
(B) The image of L is zero-dimensional.
(C) The kernel of L is one-dimensional.
(D) The image of L is one-dimensional.
(E) For any q, we have a formula above that describes a solution function that maps to q.

Your answer: ____________________________

Let n be a nonnegative integer. Denote by P_n the vector space of all polynomials in one variable x that have degree $\leq n$. P_n is a subspace of $\mathbb{R}[x]$, which in turn can be viewed as a subspace of $C^\infty(\mathbb{R})$ through the natural injective map. For convenience and completeness, define P_{-1} to be the zero subspace.

Differentiation defines a linear transformation from $C^\infty(\mathbb{R})$ to itself.

(8) What are the kernel and image of the restriction of differentiation to P_n? The result should be valid for all positive integers n.

(A) The kernel and image are both P_n
(B) The kernel is the zero subspace and the image is P_n
(C) The kernel is P_n and the image is the zero subspace
(D) The kernel is P_{n-1} and the image is P_0 (the subspace of constant functions)
(E) The kernel is P_0 and the image is P_{n-1}

Your answer: ____________________________

(9) What are the kernel and image of the restriction of differentiation to all of $\mathbb{R}[x]$?
(A) The kernel and image are both $\mathbb{R}[x]$
(B) The kernel is the zero subspace and the image is $\mathbb{R}[x]$
(C) The kernel is $\mathbb{R}[x]$ and the image is the zero subspace
(D) The kernel is $\mathbb{R}[x]$ and the image is P_0 (the subspace of constant functions)
(E) The kernel is P_0 and the image is $\mathbb{R}[x]$

Your answer: ____________________________

(10) We can use differentiation to define a linear transformation from $\mathbb{R}(x)$ to $\mathbb{R}(x)$, where we differentiate a rational function using the quotient rule for differentiation and the known rules for differentiating polynomials. What can we say about this linear transformation?
(A) The differentiation linear transformation is bijective from $\mathbb{R}(x)$ to $\mathbb{R}(x)$, i.e., every rational function is the derivative of a unique rational function.
(B) The differentiation linear transformation is injective but not surjective from $\mathbb{R}(x)$ to $\mathbb{R}(x)$, i.e., every rational function is the derivative of *at most one* rational function, but there do exist rational functions that are not expressible as the derivative of any rational function.
(C) The differentiation linear transformation is surjective but not injective from $\mathbb{R}(x)$ to $\mathbb{R}(x)$, i.e., every rational function is the derivative of *at least one* rational function, but there do exist rational functions that occur as derivatives of more than one rational function.
(D) The differentiation linear transformation is neither injective nor surjective from $\mathbb{R}(x)$ to $\mathbb{R}(x)$.

Your answer: ____________________________

(11) Denote by $\mathbb{R}[[x]]$ the vector space of all formal power series with real coefficients in one variable, i.e., series of the form:

$$\sum_{i=0}^{\infty} a_i x^i$$

Formal differentiation defines a linear transformation from $\mathbb{R}[[x]]$ to itself. What can we say about this linear transformation?
(A) The formal differentiation linear transformation is bijective from $\mathbb{R}[[x]]$ to $\mathbb{R}[[x]]$.
(B) The formal differentiation linear transformation is injective but not surjective from $\mathbb{R}[[x]]$ to $\mathbb{R}[[x]]$.
(C) The formal differentiation linear transformation is surjective but not injective from $\mathbb{R}[[x]]$ to $\mathbb{R}[[x]]$.
(D) The formal differentiation linear transformation is neither injective nor surjective from $\mathbb{R}[[x]]$ to $\mathbb{R}[[x]]$.

Your answer: ____________________________

(12) Consider the following two linear transformations $T_1, T_2 : \mathbb{R}[x] \to \mathbb{R}[x]$: T_1 is differentiation, and T_2 is multiplication by x. Which of the following is true?
(A) Both T_1 and T_2 are injective, but neither is surjective.
(B) Both T_1 and T_2 are surjective, but neither is injective.
(C) T_1 is injective but not surjective. T_2 is surjective but not injective.
(D) T_1 is surjective but not injective. T_2 is injective but not surjective.
(E) Neither T_1 nor T_2 is injective. Neither T_1 nor T_2 is surjective.

Your answer: ____________________________

(13) Consider the linear transformations T_1 and T_2 of the preceding question. What can we say regarding whether T_1 and T_2 commute?

(A) T_1 and T_2 commute.
(B) T_1 and T_2 do not commute.

Your answer: ____________________________