TAKE-HOME CLASS QUIZ: DUE FRIDAY NOVEMBER 22: LINEAR DYNAMICAL SYSTEMS

MATH 196, SECTION 57 (VIPUL NAIK)

Your name (print clearly in capital letters): ____________________

PLEASE FEEL FREE TO DISCUSS *ALL* QUESTIONS.

This quiz covers a topic that we will not be able to get to formally in the course due to time constraints. The corresponding section of the book is Section 7.1, and there is more relevant material discussed in the later sections of Chapter 7. However, you do not need to read those sections in order to attempt this quiz. Also, simply mastering the computational techniques in those sections of the book will not help you much with the quiz questions.

The questions here consider a linear dynamical system. Consider a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$. Let A be the matrix of T, so that A is a $n \times n$ matrix. For any positive integer r, the matrix A^r is the matrix for the linear transformation T^r (note here that T^r refers to the r-fold *composite* of T). The goal is to determine, starting off with an arbitrary vector $\vec{x} \in \mathbb{R}^n$, how the following sequence behaves:

$$\vec{x}, T(\vec{x}), T^2(\vec{x}), T^3(\vec{x}), \dots$$

More explicitly, each term of the sequence is obtained by applying T to the preceding term. In other words, the sequence is:

$$\vec{x}, T(\vec{x}), T(T(\vec{x})), T(T(T(\vec{x}))), \dots$$

(1) What is the necessary and sufficient condition on A such that for *every* choice of $\vec{x} \in \mathbb{R}^n$, the sequence described above eventually reaches, and stays at, the zero vector? Note that if it reaches the zero vector, it must do so in at most n steps. Please see Option (E) before answering.
(A) A is a nilpotent matrix.
(B) A is an idempotent matrix.
(C) A is an invertible matrix.
(D) A is a non-invertible matrix.
(E) None of the above.

Your answer: ____________________

(2) What is the necessary and sufficient condition on A such that there *exists* a nonzero vector $\vec{x} \in \mathbb{R}^n$ for which the sequence described above eventually reaches, and stays at, the zero vector? Note that if it reaches the zero vector, it must do so in at most n steps. Please see Option (E) before answering.
(A) A is a nilpotent matrix.
(B) A is an idempotent matrix.
(C) A is an invertible matrix.
(D) A is a non-invertible matrix.
(E) None of the above.

Your answer: ____________________

(3) What is the necessary and sufficient condition on A such that for *every* choice of $\vec{x} \in \mathbb{R}^n$, the sequence described above returns to $\vec{x}$ after a finite and positive number of steps? Please see Option (E) before answering.
(A) A is a nilpotent matrix.
(B) A is an idempotent matrix.
(C) A is an invertible matrix.

(D) A is a non-invertible matrix.
(E) None of the above.

Your answer: ____________________________

(4) What is the necessary and sufficient condition on A such that there *exists* a nonzero vector $\vec{x} \in \mathbb{R}^n$ for which the sequence described above returns to $\vec{x}$ after a finite and positive number of steps? Please see Option (E) before answering.
(A) A is a nilpotent matrix.
(B) A is an idempotent matrix.
(C) A is an invertible matrix.
(D) A is a non-invertible matrix.
(E) None of the above.

Your answer: ____________________________

(5) Suppose $n = 2$ and T is a rotation by an angle that is a rational multiple of π. What can we say about the range of the sequence

$$\vec{x}, T(\vec{x}), T^2(\vec{x}), T^3(\vec{x}), \ldots$$

starting from a nonzero vector $\vec{x}$?
(A) The range is finite, i.e., there are only finitely many distinct vectors in the sequence.
(B) The range is infinite and forms a dense subset of the circle centered at the origin and with radius equal to the length of the vector $\vec{x}$. However, it is not the entire circle.
(C) The range is infinite and is the entire circle centered at the origin and with radius equal to the length of the vector $\vec{x}$.
(D) The range is infinite and forms a dense subset of the line of the vector $\vec{x}$ (excluding the origin), but is not the entire line (excluding the origin).
(E) The range is infinite and is the entire line of the vector $\vec{x}$, excluding the origin.

Your answer: ____________________________

(6) Suppose $n = 2$ and T is a rotation by an angle that is a irrational multiple of π. What can we say about the range of the sequence

$$\vec{x}, T(\vec{x}), T^2(\vec{x}), T^3(\vec{x}), \ldots$$

starting from a nonzero vector $\vec{x}$?
(A) The range is finite, i.e., there are only finitely many distinct vectors in the sequence.
(B) The range is infinite and forms a dense subset of the circle centered at the origin and with radius equal to the length of the vector $\vec{x}$. However, it is not the entire circle.
(C) The range is infinite and is the entire circle centered at the origin and with radius equal to the length of the vector $\vec{x}$.
(D) The range is infinite and forms a dense subset of the line of the vector $\vec{x}$ (excluding the origin), but is not the entire line (excluding the origin).
(E) The range is infinite and is the entire line of the vector $\vec{x}$, excluding the origin.

Your answer: ____________________________

We return to generic n now.

(7) A nonzero vector $\vec{x}$ is termed an *eigenvector* for a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ with *eigenvalue* a real number $\lambda \in \mathbb{R}$ if $T(\vec{x}) = \lambda\vec{x}$. Note that λ is allowed to be 0. We sometimes conflate the roles of T and its matrix A, so that we call $\vec{x}$ an eigenvector for A and λ an eigenvalue for A.

If $\vec{x}$ is an eigenvector of T (or equivalently, of A) with eigenvalue λ, which of the following is true? We denote by I_n the identity transformation from $\mathbb{R}^n$ to itself.
(A) $\vec{x}$ must be in the kernel of the linear transformation $T + \lambda I_n$
(B) $\vec{x}$ must be in the image of the linear transformation $T + \lambda I_n$
(C) $\vec{x}$ must be in the kernel of the linear transformation $T - \lambda I_n$
(D) $\vec{x}$ must be in the image of the linear transformation $T - \lambda I_n$
(E) $\vec{x}$ must be in the kernel of the linear transformation λT

Your answer: ____________________________

(8) As above, let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation with matrix A. Use the terminology of eigenvector and eigenvalue from the preceding question. Which of the following is a characterization of the situation that A is a diagonal matrix?
(A) Every nonzero vector in $\mathbb{R}^n$ is an eigenvector for T.
(B) Every standard basis vector in $\mathbb{R}^n$ is an eigenvector for T.
(C) Every vector with at least one zero coordinate in $\mathbb{R}^n$ is an eigenvector for T.
(D) T has a unique eigenvector (up to scalar multiples, i.e., all eigenvectors of T are scalar multiples of each other).
(E) T has no eigenvector.

Your answer: ____________________________

(9) As above, let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation with matrix A. Use the terminology of eigenvector and eigenvalue from the preceding question. Which of the following is a characterization of the situation that A is a scalar matrix (i.e., a diagonal matrix with all diagonal entries equal)?
(A) Every nonzero vector in $\mathbb{R}^n$ is an eigenvector for T.
(B) Every standard basis vector in $\mathbb{R}^n$ is an eigenvector for T.
(C) Every vector with at least one zero coordinate in $\mathbb{R}^n$ is an eigenvector for T.
(D) T has a unique eigenvector (up to scalar multiples, i.e., all eigenvectors of T are scalar multiples of each other).
(E) T has no eigenvector.

Your answer: ____________________________

(10) Suppose A is a strictly upper-triangular $n \times n$ matrix, i.e., all entries of A that are on or below the main diagonal are zero. T is the linear transformation corresponding to A. It will turn out that the only eigenvalue for T is 0. What can we say about the eigenvectors for T for this eigenvalue?
(A) All nonzero vectors in $\mathbb{R}^n$ are eigenvectors for T with eigenvalue 0.
(B) All standard basis vectors in $\mathbb{R}^n$ are eigenvectors for T with eigenvalue 0.
(C) The vector $\vec{e}_1$ is an eigenvector for T with eigenvalue 0. The information presented is not sufficient to determine whether any of the other standard basis vectors is an eigenvector.
(D) The vector $\vec{e}_n$ is an eigenvector for T with eigenvalue 0. The information presented is not sufficient to determine whether any of the other standard basis vectors is an eigenvector.
(E) At least one of the standard basis vectors is an eigenvector for T with eigenvalue 0. However, the information presented is not sufficient to say definitively for any particular standard basis vector that it is an eigenvector.

Your answer: ____________________________

(11) Suppose A is a strictly upper-triangular $n \times n$ matrix, i.e., all entries of A that are on or below the main diagonal are zero. T is the linear transformation corresponding to A. Which of the following is A guaranteed to be? Please see Options (D) and (E) before answering.
(A) Nilpotent
(B) Idempotent
(C) Invertible
(D) All of the above

(E) None of the above

Your answer: ____________________________

(12) Consider the case $n = 2$ and let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a rotation by an angle that is *not* an *integer* multiple of π. What can we say about the set of eigenvectors and eigenvalues for T?

(A) T has no eigenvectors

(B) T has one eigenvector (up to scalar multiples) with eigenvalue 1

(C) T has one eigenvector (up to scalar multiples) and the eigenvalue depends on the angle of rotation

(D) T has two linearly independent eigenvectors (so that the set of all eigenvectors is obtained as the set of scalar multiples of either one of these vectors) with the same eigenvalue

(E) T has two linearly independent eigenvectors (so that the set of all eigenvectors is obtained as the set of scalar multiples of either one of these vectors) with distinct eigenvalues

Your answer: ____________________________