TAKE-HOME CLASS QUIZ: DUE FRIDAY NOVEMBER 15: IMAGE AND KERNEL

MATH 196, SECTION 57 (VIPUL NAIK)

Your name (print clearly in capital letters): ____________________________________

PLEASE FEEL FREE TO DISCUSS *ALL* QUESTIONS.

The purpose of this quiz is to review in greater depth the ideas behind image and kernel. The goal of the first seven questions is to review the ideas of injectivity, surjectivity, and bijectivity in the context of arbitrary functions between sets. The purpose is two-fold: (i) to give a functions-based approach to justifying, intuitively and formally, facts about the effect of matrix multiplication on rank, and (ii) to hint at ways in which linear transformations behave better than other types of functions.

The corresponding lecture notes are titled `Image and kernel of a linear transformation` and the corresponding section of the text is Section 3.1.

Just as a reminder, a function $f : A \to B$ between sets A and B is said to be:

- *injective* if for every $b \in B$, there is *at most* one value of a such that $f(a) = b$. In other words, if we denote by $f^{-1}(b)$ the set $\{a \in A \mid f(a) = b\}$, then $|f^{-1}(b)| \leq 1$ for all $b \in B$ (here $|f^{-1}(b)|$ denotes the size of the set $f^{-1}(b)$).
- *surjective* if for every $b \in B$, there is *at least* one value of a such that $f(a) = b$. In other words, if we denote by $f^{-1}(b)$ the set $\{a \in A \mid f(a) = b\}$, then $|f^{-1}(b)| \geq 1$ for all $b \in B$.
- *bijective* if for every $b \in B$, there is *exactly* one value of a such that $f(a) = b$. In other words, if we denote by $f^{-1}(b)$ the set $\{a \in A \mid f(a) = b\}$, then $|f^{-1}(b)| = 1$ for all $b \in B$.

(1) Suppose $g : A \to B$ and $f : B \to C$ are functions. The composite $f \circ g$ is a function from A to C. What can we say the relationship between the injectivity of $f \circ g$, the injectivity of f, and the injectivity of g?
 (A) $f \circ g$ is injective if and only if f and g are both injective.
 (B) If f and g are both injective, then $f \circ g$ is injective. However, $f \circ g$ being injective does not imply anything about the injectivity of either f or g.
 (C) If f and g are both injective, then $f \circ g$ is injective. If $f \circ g$ is injective, then at least one of f and g is injective, but we cannot conclusively say for any specific one of the two that it must be injective.
 (D) If f and g are both injective, then $f \circ g$ is injective. If $f \circ g$ is injective, then f is injective, but we do not have enough information to deduce whether g is injective.
 (E) If f and g are both injective, then $f \circ g$ is injective. If $f \circ g$ is injective, then g is injective, but we do not have enough information to deduce whether f is injective.

 Your answer: ____________________________

(2) Suppose $g : A \to B$ and $f : B \to C$ are functions. The composite $f \circ g$ is a function from A to C. What can we say the relationship between the surjectivity of $f \circ g$, the surjectivity of f, and the surjectivity of g?
 (A) $f \circ g$ is surjective if and only if f and g are both surjective.
 (B) If f and g are both surjective, then $f \circ g$ is surjective. However, $f \circ g$ being surjective does not imply anything about the surjectivity of either f or g.
 (C) If f and g are both surjective, then $f \circ g$ is surjective. If $f \circ g$ is surjective, then at least one of f and g is surjective, but we cannot conclusively say for any specific one of the two that it must be surjective.
 (D) If f and g are both surjective, then $f \circ g$ is surjective. If $f \circ g$ is surjective, then f is surjective, but we do not have enough information to deduce whether g is surjective.

(E) If f and g are both surjective, then $f \circ g$ is surjective. If $f \circ g$ is surjective, then g is surjective, but we do not have enough information to deduce whether f is surjective.

Your answer: ____________________________

(3) Suppose $g : A \to B$ and $f : B \to C$ are functions. The composite $f \circ g$ is a function from A to C. Suppose $f \circ g$ is bijective. What can we say about f and g individually?
(A) Both f and g must be bijective.
(B) Both f and g must be injective, but neither of them need be surjective.
(C) Both f and g must be surjective, but neither of them need be injective.
(D) f must be injective but need not be surjective. g must be surjective but need not be injective.
(E) f must be surjective but need not be injective. g must be injective but need not be surjective.

Your answer: ____________________________

(4) $g : A \to B$ and $f : B \to C$ are functions. The composite $f \circ g$ is a function from A to C. Suppose both f and g are surjective. Further, suppose that for every $b \in B$, $g^{-1}(b)$ has size m (for a fixed positive integer m) and for every $c \in C$, $f^{-1}(c)$ has size n (for a fixed positive integer n). Then, what can we say about the sizes of the fibers (i.e., the inverse images of points in C) under the composite $f \circ g$?
(A) The size is $\min\{m, n\}$
(B) The size is $\max\{m, n\}$
(C) The size is $m + n$
(D) The size is mn
(E) The size is m^n

Your answer: ____________________________

(5) **PLEASE READ THIS VERY CAREFULLY AND CONSIDER A WIDE VARIETY OF POLYNOMIAL EXAMPLES**: Suppose f is a polynomial function of degree $n > 2$ from $\mathbb{R}$ to $\mathbb{R}$. What can we say about the fibers of f, i.e., the sets of the form $f^{-1}(x)$, $x \in \mathbb{R}$?

Hint: At the one extreme, consider a polynomial of the form x^n. Consider the sizes of the fibers $f^{-1}(0)$ and $f^{-1}(x)$ for a positive value of x (the fiber size for the latter will depend on whether n is even or odd). Alternatively, consider a polynomial of the form $(x-1)(x-2)\dots(x-n)$. Consider the size of the fiber $f^{-1}(0)$.
(A) Every fiber has size n.
(B) The minimum of the sizes of fibers is exactly n, but every fiber need not have size n.
(C) The maximum of the sizes of fibers is exactly n, but every fiber need not have size n.
(D) The minimum of the sizes of fibers is at least n, but need not be exactly n.
(E) The maximum of the sizes of fibers is at most n, but need not be exactly n.

Your answer: ____________________________

(6) Suppose f is a continuous injective function from $\mathbb{R}$ to $\mathbb{R}$. What can we say about the nature of f?
(A) f must be an increasing function on all of $\mathbb{R}$.
(B) f must be a decreasing function on all of $\mathbb{R}$.
(C) f must be a constant function on all of $\mathbb{R}$.
(D) f must be either an increasing function on all of $\mathbb{R}$ or a decreasing function on all of $\mathbb{R}$, but the information presented is insufficient to decide which case occurs.
(E) f must be either an increasing function or a decreasing function or a constant function on all of $\mathbb{R}$, but the information presented is insufficient for deciding anything stronger.

Your answer: ____________________________

(7) **PLEASE READ THIS CAREFULLY, MAKE CASES, AND CHECK YOUR REASONING**: Suppose f, g, and h are continuous bijective functions from $\mathbb{R}$ to $\mathbb{R}$. What can we say about the functions $f + g$, $f + h$, and $g + h$?

Hint: Based on the preceding question, you know something about the nature of f, g, and h individually as functions, but there is some degree of ambiguity in your knowledge. Make cases based on the possibilities and see what you can deduce in the best and worst case.

(A) They are all continuous bijective functions from $\mathbb{R}$ to $\mathbb{R}$.
(B) At least two of them are continuous bijective functions from $\mathbb{R}$ to $\mathbb{R}$. However, we cannot say more.
(C) At least one of them is a continuous bijective function from $\mathbb{R}$ to $\mathbb{R}$. However, we cannot say more.
(D) Either all three sums are continuous bijective functions from $\mathbb{R}$ to $\mathbb{R}$, or none is.
(E) It is possible that none of the sums is a continuous bijective functions from $\mathbb{R}$ to $\mathbb{R}$; it is also possible that one, two, or all the sums are continuous bijective functions from $\mathbb{R}$ to $\mathbb{R}$.

Your answer: ____________________________

The questions that follow tripped up students quite a bit last time, so I urge you to proceed with caution. You can do each of these questions in either of two ways:

- Using abstract, general reasoning.
- Constructing concrete examples.

While the former approach is one you should eventually be able to embrace without trepidation, feel free to rely on the latter approach for now. For this, consider matrices describing the linear transformations and use matrix multiplication to compute the composite where needed. Compute the kernel, image, and rank using the methods known to you. Take matrices such as those arising from finite state automata (as described in the "linear transformations and finite state automata" quiz) or their generalizations to rectangular matrices.

For instance, you might try taking a matrix such as $\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$. This describes a linear transformation $\mathbb{R}^5 \to \mathbb{R}^4$ and has rank three. The dimension of the kernel (inside $\mathbb{R}^5$) is 2 (explicitly, the kernel is precisely the set of vectors in $\mathbb{R}^5$ whose first three coordinates are zero) and the dimension of the image (inside $\mathbb{R}^4$) is 3 (explicitly, the image is precisely the set of vectors in $\mathbb{R}^4$ whose fourth coordinate is 0).

(8) *This is the analogue for linear transformations of Question 1*: Suppose m, n, p are positive integers. Suppose A is a $m \times n$ matrix and B is a $n \times p$ matrix. The product AB is a $m \times p$ matrix. Denote by T_A, T_B, and T_{AB} respectively the linear transformations corresponding to A, B, and AB. We have $T_A : \mathbb{R}^n \to \mathbb{R}^m$, $T_B : \mathbb{R}^p \to \mathbb{R}^n$, and $T_{AB} : \mathbb{R}^p \to \mathbb{R}^m$. Note that $T_{AB} = T_A \circ T_B$.

Recall that a matrix has full column rank if and only if the corresponding linear transformation is injective.

Which of the following describes correctly the relationship between A having full column rank (i.e., rank n), B having full column rank (i.e., rank p), and AB having full column rank (i.e., rank p)?

(A) AB has full column rank (i.e., rank p) if and only if A and B both have full column rank (ranks n and p respectively).
(B) If A and B both have full column rank, then AB has full column rank. However, AB having full column rank does not imply anything (separately or jointly) regarding whether A or B has full column rank.
(C) If A and B both have full column rank, then AB has full column rank. If AB has full column rank, then at least one of A and B has full column rank, but we cannot definitively say for any particular one of A and B that it must have full column rank.

(D) If A and B both have full column rank, then AB has full column rank. AB having full column rank implies that A has full column rank, but it does not tell us for sure that B has full column rank.
(E) If A and B both have full column rank, then AB has full column rank. AB having full column rank implies that B has full column rank, but it does not tell us for sure that A has full column rank.

Your answer: ____________________________

(9) *This is the analogue for linear transformations of Question 2*: Suppose m, n, p are positive integers. Suppose A is a $m \times n$ matrix and B is a $n \times p$ matrix. The product AB is a $m \times p$ matrix. Denote by T_A, T_B, and T_{AB} respectively the linear transformations corresponding to A, B, and AB. We have $T_A : \mathbb{R}^n \to \mathbb{R}^m$, $T_B : \mathbb{R}^p \to \mathbb{R}^n$, and $T_{AB} : \mathbb{R}^p \to \mathbb{R}^m$. Note that $T_{AB} = T_A \circ T_B$.

Recall that a matrix has full row rank if and only if the corresponding linear transformation is surjective.

Which of the following describes correctly the relationship between A having full row rank (i.e., rank m), B having full row rank (i.e., rank n), and AB having full row rank (i.e., rank m)?
(A) AB has full row rank if and only if A and B both have full row rank.
(B) If A and B both have full row rank, then AB has full row rank. However, AB having full row rank does not imply anything (separately or jointly) regarding whether A or B has full row rank.
(C) If A and B both have full row rank, then AB has full row rank. If AB has full row rank, then at least one of A and B has full row rank, but we cannot definitively say for any particular one of A and B that it must have full row rank.
(D) If A and B both have full row rank, then AB has full row rank. AB having full row rank implies that A has full row rank, but it does not tell us for sure that B has full row rank.
(E) If A and B both have full row rank, then AB has full row rank. AB having full row rank implies that B has full row rank, but it does not tell us for sure that A has full row rank.

Your answer: ____________________________

(10) *This is the analogue for linear transformations of Question 3*: Suppose m and n are positive integers. Suppose A is a $m \times n$ matrix and B is a $n \times m$ matrix. The product AB is a $m \times m$ matrix. The corresponding linear transformations are $T_A : \mathbb{R}^n \to \mathbb{R}^m$, $T_B : \mathbb{R}^m \to \mathbb{R}^n$, and $T_{AB} : \mathbb{R}^m \to \mathbb{R}^m$.

Suppose the square matrix AB has full rank m. What can we deduce about the ranks of A and B?
(A) Both A and B have full row rank, *and* both A and B have full column rank.
(B) Both A and B have full column rank, but neither of them need have full row rank.
(C) Both A and B have full row rank, but neither of them need have full column rank.
(D) A must have full column rank but need not have full row rank. B must have full row rank but need not have full column rank.
(E) A must have full row rank but need not have full column rank. B must have full column rank but need not have full row rank.

Your answer: ____________________________

For the coming questions, we will denote vector spaces by letters such as U, V, and W. You can, however, consider them to be finite-dimensional vector spaces of the form $\mathbb{R}^n$. However, you should take care not to use a letter for the dimension of a vector space if the letter is already in use elsewhere in the question. Also, you should take care to use different letters for the dimensions of different vector spaces, unless it is given to you that the vector spaces have the same dimension. The results also hold for infinite-dimensional vector spaces, but you can work on all the problems assuming you are working in the finite-dimensional setting.

(11) *This is an analogue for linear transformations of Question 4*: Suppose $T_1 : U \to V$ and $T_2 : V \to W$ are linear transformations. The composite $T_2 \circ T_1$ is also a linear transformation, this time from U to W. Suppose the kernel of T_1 has dimension m and the kernel of T_2 has dimension n. Suppose both T_1 and T_2 are surjective. What can you say about the dimension of the kernel of $T_2 \circ T_1$?

Please note this carefully: Although this question is analogous to Question 4, the correct answer options differ for the two questions. Here is an intuitive explanation for the relationship between the questions. Question 4 asked abot the *sizes* of the fibers. This question asks about the dimensions of the kernels. The fibers do correspond to the kernels. But the relationship between dimension and size is of a *logarithmic nature*. What we mean is that the dimension can be thought of as the logarithm of the size. This isn't literally true, because the size is infinite. But metaphorically, it makes sense, because, for instance, the dimension of $\mathbb{R}^p$ is the exponent p, and that comports with the laws of logarithms (similar to how the $\log_2(2^p) = p$).

(A) The dimension is $\min\{m, n\}$.
(B) The dimension is $\max\{m, n\}$.
(C) The dimension is $m + n$.
(D) The dimension is mn.
(E) The dimension is m^n.

Your answer: ____________________________

(12) Suppose $T_1 : U \to V$ and $T_2 : V \to W$ are linear transformations. The composite $T_2 \circ T_1$ is also a linear transformation, this time from U to W. Suppose the kernel of T_1 has dimension m and the kernel of T_2 has dimension n. However, unlike the preceding question, we are not given any information about the surjectivity of either T_1 or T_2. The answer to the preceding question gives an (inclusive) *upper* bound on the dimension of the kernel of $T_2 \circ T_1$. Which of the following is the best *lower* bound we can manage in general?

(A) $|m - n|$
(B) m
(C) n
(D) $m + n$

Your answer: ____________________________

(13) Suppose $T_1, T_2 : U \to V$ are linear transformations. Which of the following is true? Please see Options (D) and (E) before answering and select the single option that best reflects your view.

(A) If both T_1 and T_2 are injective, then $T_1 + T_2$ is injective.
(B) If both T_1 and T_2 are surjective, then $T_1 + T_2$ is surjective.
(C) If both T_1 and T_2 are bijective, then $T_1 + T_2$ is bijective.
(D) All of the above
(E) None of the above

Your answer: ____________________________

(14) Suppose $T_1, T_2 : U \to V$ are linear transformations. Which of the following best describes the relation between the kernels of T_1, T_2, and $T_1 + T_2$?

(A) The kernel of $T_1 + T_2$ equals the intersection of the kernel of T_1 and the kernel of T_2.
(B) The kernel of $T_1 + T_2$ is contained inside the intersection of the kernel of T_1 and the kernel of T_2, but need not be equal to the intersection.
(C) The kernel of $T_1 + T_2$ contains the intersection of the kernel of T_1 and the kernel of T_2, but need not be equal to the intersection.
(D) The kernel of $T_1 + T_2$ is contained inside the sum of the kernel of T_1 and the kernel of T_2, but need not be equal to the sum.
(E) The kernel of $T_1 + T_2$ contains the sum of the kernel of T_1 and the kernel of T_2, but need not be equal to the sum.

Your answer: ____________________________

(15) Suppose $T_1, T_2 : U \to V$ are linear transformations. Which of the following best describes the relation between the images of T_1, T_2, and $T_1 + T_2$?
(A) The image of $T_1 + T_2$ equals the intersection of the image of T_1 and the image of T_2.
(B) The image of $T_1 + T_2$ is contained inside the intersection of the image of T_1 and the image of T_2, but need not be equal to the intersection.
(C) The image of $T_1 + T_2$ contains the intersection of the image of T_1 and the image of T_2, but need not be equal to the intersection.
(D) The image of $T_1 + T_2$ is contained inside the sum of the image of T_1 and the image of T_2, but need not be equal to the sum.
(E) The image of $T_1 + T_2$ contains the sum of the image of T_1 and the image of T_2, but need not be equal to the sum.

Your answer: ____________________________

(16) Suppose T is a linear transformation from a vector space V to itself. Note that V may be an infinite-dimensional space, such as $C^\infty(\mathbb{R})$ (with T being differentiation), but for convenience, you can imagine V to be finite-dimensional (we will not reference the dimension of V in this question, however). Suppose the kernel of T has dimension n. What can you say from this information about the dimension of the kernel of T^r for a positive integer r?
(A) It is at least n and at most $n + r$.
(B) It is at least n and at most nr.
(C) It is at least $n + r$ and at most nr.
(D) It is at least $n + r$ and at most n^r.

Your answer: ____________________________

The next few questions deal with the relationship between the rows and columns of the matrix on the one hand, and the image and kernel of the linear transformation on the other hand.

(17) Suppose A is a $n \times m$ matrix and $T_A : \mathbb{R}^m \to \mathbb{R}^n$ is the corresponding linear transformation. Which of the following correctly describes the relationship between the rows and columns of A and the image and kernel of T_A?
(A) The kernel of T_A is precisely the subspace of $\mathbb{R}^m$ spanned by the rows of A. The image of T_A is precisely the subspace of $\mathbb{R}^n$ spanned by the columns of A.
(B) The kernel of T_A is precisely the subspace of $\mathbb{R}^m$ spanned by the columns of A. The image of T_A is precisely the subspace of $\mathbb{R}^n$ spanned by the rows of A.
(C) The kernel of T_A is precisely the subspace of $\mathbb{R}^m$ comprising the vectors that are *orthogonal* to the rows of A. The image of T_A is precisely the subspace of $\mathbb{R}^n$ comprising the vectors that are *orthogonal* to the columns of A.
(D) The kernel of T_A is precisely the subspace of $\mathbb{R}^m$ comprising the vectors that are *orthogonal* to the rows of A. The image of T_A is the subspace of $\mathbb{R}^n$ spanned by the columns of A.
(E) The kernel of T_A is precisely the subspace of $\mathbb{R}^m$ spanned by the rows of A. The image of T_A is precisely the subspace of $\mathbb{R}^n$ comprising the vectors that are *orthogonal* to the columns of A.

Your answer: ____________________________

(18) Suppose A and B are $n \times m$ matrices, $T_A : \mathbb{R}^m \to \mathbb{R}^n$ is the linear transformation corresponding to A, and $T_B : \mathbb{R}^m \to \mathbb{R}^n$ is the linear transformation corresponding to B. Which of the following correctly describes the relation between the rows, columns, image and kernel? Please see Option (E) before answering.
(A) If B can be obtained from A by a sequence of row interchange operations, then T_A and T_B have the same kernel as each other and also the same image as each other.
(B) If B can be obtained from A by a sequence of column interchange operations, then T_A and T_B have the same kernel as each other and also the same image as each other.

(C) If B can be obtained from A by a sequence of row interchange operations, then T_A and T_B have the same kernel as each other. If B can be obtained from A by a sequence of column interchange operations, then T_A and T_B have the same image as each other.

(D) If B can be obtained from A by a sequence of row interchange operations, then T_A and T_B have the same image as each other. If B can be obtained from A by a sequence of column interchange operations, then T_A and T_B have the same kernel as each other.

(E) None of the above.

Your answer: ____________________________