# REVIEW SHEET FOR MIDTERM 2: BASIC

MATH 195, SECTION 59 (VIPUL NAIK)

**To maximize efficiency, please bring a copy (print or readable electronic) of this review sheet to the review session.**

*The document does not include material that was part of the midterm 1 syllabus. Very little of that material will appear directly in midterm 2; however, you should have reasonable familiarity with the material.*

We will not be going over this sheet, but rather, we'll be going over the advanced review sheet in the session. Please review this sheet on your own time.

## 1. Formula summary

### 1.1. Formula formulas.

(1) Unit vectors parallel to a nonzero vector $v$ are $v/|v|$ and $-v/|v|$.

(2) Coordinates of the unit vector are the direction cosines. If $v/|v| = \langle \ell, m, n \rangle$, these are the direction cosines. If $\alpha, \beta, \gamma \in [0, \pi]$ are such that $\cos\alpha = \ell$, $\cos\beta = m$, $\cos\gamma = n$, then $\alpha, \beta, \gamma$ are the direction angles.

(3) Parametric equation of line in $\mathbb{R}^3$: $\mathbf{r} = \mathbf{r_0} + t\mathbf{v}$, $\mathbf{r_0}$ is the radial vector for a point in the line, $\mathbf{v}$ is the difference vector between two points in the line. In scalar terms, $x = x_0 + ta$, $y = y_0 + tb$, $z = z_0 + tc$, where $\mathbf{r} = \langle x, y, z \rangle$, $\mathbf{r_0} = \langle x_0, y_0, z_0 \rangle$, and $\mathbf{v} = \langle a, b, c \rangle$. (See also two-point form parametric equation).

(4) Symmetric equation of line in $\mathbb{R}^3$ not parallel to any coordinate plane (i.e., $abc \neq 0$ case):

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

with same notation as for parametric equation. (See also cases of parallel to coordinate plane).

(5) Equation of plane: vector equation $\mathbf{n} \cdot \mathbf{r} = \mathbf{n} \cdot \mathbf{r_0}$ where $\mathbf{n}$ is a nonzero normal vector, $\mathbf{r_0}$ is a fixed point in the plane. If $\mathbf{n} = \langle a, b, c \rangle$, $\mathbf{r_0} = \langle x_0, y_0, z_0 \rangle$, and $\mathbf{r} = \langle x, y, z \rangle$, we get:

$$ax + by + cz = ax_0 + by_0 + cz_0$$

(6) For a function $z = f(x, y)$, the tangent plane to the graph of this function (a surface in $\mathbb{R}^3$) at the point $(x_0, y_0, f(x_0, y_0))$ *such that $f$ is differentiable at the point* $(x_0, y_0)$ is the plane:

$$z - f(x_0, y_0) = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

The corresponding linear function we get is:

$$L(x, y) = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

This provides a linear approximation to the function near the point where we are computing the tangent plane.

### 1.2. Artistic formulas.

(1) Partial differentiation, multiplicatively separable – differentiate each piece in the corresponding variable the corresponding number of times.

(2) Partial differentiation, additively separable – pure partials, just care about function of that variable, mixed partials are zero.

(3) Integration along rectangular region, multiplicatively separable – product of integrals for function of each variable.

(4) Integration along non-rectangular region, multiplicatively separable – outer variable function can be pulled to outer integral.

## 2. Equations of lines and planes

### 2.1. **Direction cosines.**

(1) For a nonzero vector $v$, there are two unit vectors parallel to $v$, namely $v/|v|$ and $-v/|v|$.
(2) The direction cosines of $v$ are the coordinates of $v/|v|$. if $v/|v| = \langle \ell, m, n\rangle$, then the direction cosines are $\ell$, $m$, and $n$. We have the relation $\ell^2 + m^2 + n^2 = 1$. Further, if $\alpha$, $\beta$, and $\gamma$ are the angles made by $v$ with the positive $x$, $y$, and $z$ axes, then $\ell = \cos\alpha$, $m = \cos\beta$, and $n = \cos\gamma$.

### 2.2. **Lines.** Words ...

(1) A line in $\mathbb{R}^3$ has dimension 1 and codimension 2. A parametric description of a line thus requires 1 parameter. A top-down equational description requires two equations.
(2) Given a point with radial vector $\mathbf{r}_0$ and a direction vector $\mathbf{v}$ along a line, the parametric description of the line is given by $\mathbf{r}(t) = \mathbf{r}_0 + t\mathbf{v}$. If $\mathbf{r}_0 = \langle x_0, y_0, z_0\rangle$ and $\mathbf{v} = \langle a, b, c\rangle$, this is more explicitly described as $x = x_0 + ta$, $y = y_0 + tb$, $z = z_0 + tc$.
(3) Given two points with radial vectors $\mathbf{r_0}$ and $\mathbf{r_1}$, we obtain a vector equation for the line as $\mathbf{r}(t) = t\mathbf{r_1} + (1-t)\mathbf{r_0}$. If we restrict $t$ to the interval $[0,1]$, then we get the line segment joining the points with these radial vectors.
(4) If the line is not parallel to any of the coordinate planes, this parametric description can be converted to symmetric equations by eliminating the parameter $t$. With the above notation, we get:

$$\frac{x-x_0}{a} = \frac{y-y_0}{b} = \frac{z-z_0}{c}$$

This is actually *two* equations rolled into one.
(5) If $c = 0$ and $ab \neq 0$, the line is parallel to the $xy$-plane, and we get the equations:

$$\frac{x-x_0}{a} = \frac{y-y_0}{b}, \qquad z = z_0$$

Similarly for the other cases where precisely one coordinate is zero.
(6) If $a = b = 0$ and $c \neq 0$, the line is parallel to the $z$-axis, and we get the equations:

$$x = x_0, \qquad y = y_0$$

Actions ...

(1) To intersect two lines both given parametrically: Choose different letters for parameters, equate coordinates, solve 3 equations in 2 variables. *Note: Expected dimension of solution space is* $2 - 3 = -1$.
(2) To intersect a line given parametrically and a line given by equations: Plug in the coordinates as functions of parameters into both equations, solve. Solve 2 equations in 1 variable. *Note: Expected dimension of solution space is* $1 - 2 = -1$.
(3) To intersect two lines given by equations: Combine equations, solve 4 equations in 3 variables. *Note: Expected dimension of solution space is* $3 - 4 = -1$.

### 2.3. **Planes.** Words ...

(1) Vector equation of a plane (for the radial vector) is $\mathbf{n}\cdot(\mathbf{r} - \mathbf{r_0}) = 0$ where $\mathbf{n}$ is a normal vector to the plane and $\mathbf{r_0}$ is the radial vector of any fixed point in the plane. This can be rewritten as $\mathbf{n}\cdot\mathbf{r} = \mathbf{n}\cdot\mathbf{r_0}$. Using $\mathbf{n} = \langle a, b, c\rangle$, $\mathbf{r} = \langle x, y, z\rangle$, and $\mathbf{r}_0 = \langle x_0, y_0, z_0\rangle$, we get the corresponding scalar equation $ax + by + cz = ax_0 + by_0 + cz_0$. Set $d = -(ax_0 + by_0 + cz_0)$ and we get $ax + by + cz + d = 0$.
(2) The "direction" or "parallel family" of a plane is determined by its normal vector. The angle between planes is the angle between their normal vectors. Two planes are parallel if their normal vecors are parallel. And so on.

Actions ...

(1) Given three non-collinear points, we find the equation of the unique plane containing them as follows: first we find a normal vector by taking the cross product of two of the difference vectors. Then we use any of the three points to calculate the dot product with the normal vector in the above vector equation or the corresponding scalar equation.

Note that if the points are collinear, there is no unique plane through them – any plane containing their line is a plane containing them.

(2) We can compute the angle of intersection of two planes by computing the angle of intersection of their normal vectors.
(3) The line of intersection of two planes that are not parallel can be computed by simply taking the equations for *both* planes. This, however, is not a standard form for a line in $\mathbb{R}^3$. To find a standard form, either find two points by inspection and join them, or find one point by inspection and another point by taking the cross product of the normal vectors to the plane.
(4) To intersect a plane and a line, plug in parametric expressions for the coordinates arising from the line into the equation of the plane. We get one equation in the one parameter variable. In general, this is expected to have a unique solution for the parameter. Plug in the value of the parameter into the parametric expressions for the line and get the coordinates of the point of intersection.
(5) For a point with coordinates $(x_1, y_1, z_1)$ and a plane $ax + by + cz + d = 0$, the distance of the point from the plane is given by $|ax_1 + by_1 + cz_1 + d|/\sqrt{a^2 + b^2 + c^2}$.

## 3. Functions of several variables

### 3.1. Introduction.

Words ...

(1) A function of $n$ variables is a function on a subset of $\mathbb{R}^n$. We can think of it in three ways: as a function with $n$ real inputs, as a function with input a point in (a subset of) $\mathbb{R}^n$, and as a function with $n$-dimensional vector inputs. We often write the inputs with numerical subscripts, so a function $f$ of $n$ inputs is written as $f(x_1, x_2, \dots, x_n)$.
(2) In the case $n = 2$, we often write the inputs as $x, y$ so we write $f(x, y)$. This may be concretely described as an expression in terms of $x$ and $y$.
(3) The graph of a function $f(x, y)$ of the two variables $x$ and $y$ is the surface $z = f(x, y)$. The $xy$-plane plays the role of the independent variable plane and the $z$-axis is the dependent variable axis. Any such graph satisfies the "vertical" line test where vertical means parallel to the $z$-axis.
(4) The level curves of a function $f(x, y)$ are curves satisfying $f(x, y) = z_0$ for some fixed $z_0$. These are curves in the $xy$-plane.
(5) The level surfaces of a function $f(x, y, z)$ of three variables are the surfaces satisfying $f(x, y, z) = c$ for some fixed $c$.
(6) Domain convention: If nothing else is specified, the domain of a function in $n$ variables given by an expression is defined as the largest subset of $\mathbb{R}^n$ on which that expression makes sense.
(7) We can also define vector-valued functions of many variables, e.g., a function from a subset of $\mathbb{R}^m$ to a subset of $\mathbb{R}^n$.
(8) We can do various pointwise combination operations on functions of many variables, similar to what we do for functions of one variable (both the scalar and vector cases).
(9) To compose functions, we need that the number of outputs of the inner/right function equals the number of inputs of the outer/left function.

Actions ...

(1) To find the domain, we first apply the usual conditions on denominators, things under square roots, and inputs to logarithms and inverse trigonometric functions. For functions of two variables, each such condition usually gives a region of $\mathbb{R}^2$ bounded by a line or curve.
(2) After getting a bunch of conditions that need to be satisfied, we try to find the common solution set for all of these. This involves intersecting the regions in $\mathbb{R}^2$ obtained previously.

### 3.2. Limits and continuity.

Words ...

(1) Conceptual definition of limit $\lim_{x \to c} f(x) = L$: For any neighborhood of $L$, however small, there exists a neighborhood of $c$ such that for all $x \neq c$ in that neighborhood of $c$, $f(x)$ is in the original neighborhood of $L$.
(2) Other conceptual definition of limit $\lim_{x \to c} f(x) = L$: For any open ball centered at $L$, however small, there exists an open ball centered at $c$ such that for all $x \neq c$ in that open ball, $f(x)$ lies in the original open ball centered at $L$.

(3) $\epsilon - \delta$ definition of limit $\lim_{x \to c} f(x) = L$: For any $\epsilon > 0$, there exists $\delta > 0$ such that for all $x = \langle x_1, x_2, \dots, x_n \rangle$ satisfying $0 < |x - c| < \delta$, we have $|f(x) - L| < \epsilon$. The definition is the same for vector inputs and vector outputs, but we interpret subtraction as vector subtraction and we interpret $|\cdot|$ as length/norm of a vector rather than absolute value if dealing with vectors instead of scalars.
(4) On the range/image side, it is possible to break down continuity into continuity of each component, i.e., a vector-valued function is continuous if each component scalar function is continuous. This cannot be done on the domain side.
(5) We can use the above definition of limit to define a notion of continuity. The usual limit theorems and continuity theorems apply.
(6) The above definition of continuity, when applied to functions of many variables, is termed *joint continuity*. For a jointly continuous function, the restriction to any continuous curve is continuous with respect to the parameterization.
(7) We can define a function of many variables to be a continuous in a particular variable if it is continuous in that variable when we fix the values of all other variables. A function continuous in each of its variables is termed *separately continuous*. Any jointly continuous function is separately continuous, but the converse is not necessarily true.
(8) Geometrically, separate continuity means continuity along directions parallel to the coordinate axes.
(9) For homogeneous functions, we can talk of the order of a zero at the origin by converting to radial/polar coordinates and then seeing the order of the zero in terms of $r$.

Actions ...

(1) Polynomials and sin and cos are continuous, and things obtained by composing/combining these are continuous. Rational functions are continuous wherever the denominator does not blow up. The usual *plug in to find the limit* rule, as well as the usual list of indeterminate forms, applies.
(2) Unlike the case of functions of one variable, the strategy of canceling common factors is not sufficient to calculate all limits for rational functions. When this fails, and we need to compute a limit at the origin, try doing a polar coordinates substitution, i.e., $x = r\cos\theta$, $y = r\sin\theta$, $r > 0$. Now try to find the limit as $r \to 0$. If you get an answer independent of $\theta$ in a strong sense, then that's the limit. This method works best for homogeneous functions.
(3) For limit computations, we can use the usual chaining and stripping techniques developed for functions of one variable.

3.3. **Partial derivatives.** Words ...

(1) The partial derivative of a function of many variables with respect to any one variable is the derivative with respect to that variable, keeping others constant. It can be written as a limit of a difference quotient, using variable letter subscript (such as $f_x(x, y)$), numerical subscript based on input position (such as $f_2(x_1, x_2, x_3)$), Leibniz notation (such as $\partial/\partial x$).
(2) In the separate continuity-joint continuity paradigm, partial derivatives correspond to the "separate" side. The corresponding "joint" side notion requires linear algebra and we will therefore defer it.
(3) The expression for the partial derivative of a function of many variables with respect to any one of them involves all the variables, not just the one being differentiated against (the exception is additively separable functions). In particular, the *value* of the partial derivative (as a number) depends on the values of all the inputs.
(4) The procedure for partial derivatives differs from the procedure used for implicit differentiation: in partial derivatives, we assume that the other variable is independent and constant, while in implicit differentiation, we treat the other variable as an unknown (implicit) function of the variable.
(5) We can combine partial derivatives and implicit differentiation, for instance, $G(x, y, z) = 0$ may be a description of $z$ as an implicit function of $x$ and $y$, and we can compute $\partial z/\partial x$ by implicit differentiation, differentiate $G$, treat $z$ as an implicit function of $x$ and treat $y$ as a constant.
(6) By iterating partial differentiation, we can define higher order partial derivatives. For instance $f_{xx}$ is the derivative of $f_x$ with respect to $x$. For a function of two variables $x$ and $y$, we have four second order partials: $f_{xx}$, $f_{yy}$, $f_{xy}$ and $f_{yx}$.
(7) Clairaut's theorem states that if $f$ is defined in an open disk surrounding a point, and both mixed partials $f_{xy}$ and $f_{yx}$ are jointly continuous in the open disk, then $f_{xy} = f_{yx}$ at the point.

(8) We can take higher order partial derivatives. By iterated application of Clairaut's theorem, we can conclude that under suitable continuity assumptions, the mixed partials having the same number of differentiations with respect to each variable are equal in value.
(9) We can consider a partial differential equation for functions of many variables. This is an equation involving the function and its partial derivatives (first or higher order) all at one point. A solution is a function of many variables that, when plugged in, satisfies the partial differential equation.
(10) Unlike the case of ordinary differential equations, the solution spaces to partial differential equations are huge, usually infinite-dimensional, and there is often no neat description of the general solution.

Pictures ...

(1) The partial derivatives can be interpreted as slopes of tangent lines to graphs of functions of the one variable being differentiated with respect to, once we fix the value of the other variable.

Actions ...

(1) To compute the first partials, differentiate with respect to the relevant variable, treating other variables as constants.
(2) Implicit differentiation for first partial of implicit function of two variables, e.g., $z$ as a function of $x$ and $y$ given via $G(x, y, z) = 0$.
(3) In cases where differentiation formulas do not apply directly, use the limit of difference quotient idea.
(4) To calculate partial derivative at a point, it may be helpful to first fix the values of the other coordinates and then differentiate the function of one variable rather than trying to compute the general expression for the derivative using partial differentiation and then plugging in values. On the other hand, it might not.
(5) Two cases of particular note for computing partial derivatives are the cases of additively and multiplicatively separable functions.
(6) To find whether a function satisfies a partial differential equation, plug it in and check. Don't try to find a general solution to the partial differential equation.

Econ-speak ...

(1) Partial derivatives = marginal analysis. Positive = increasing, negative = decreasing
(2) Second partial derivatives = nature of returns to scale. Positive = increasing returns (concave up), zero = constant returns (linear), negative = decreasing returns (concave down)
(3) Mixed partial derivatives = interaction analysis; positive mixed partial derivative means complementary, negative mixed partial derivative means substitution
(4) The signs of the first partials are invariant under monotone transformations, not true for signs of second partials, pure or mixed.
(5) Examples of quantity demanded, production functions.
(6) Cobb-Douglas production functions (see section of lecture notes and corresponding discussion in the book)

3.4. **Tangent planes and linear approximations.** Words ...

(1) For a $d$-dimensional subset of $\mathbb{R}^n$, it (occasionally) makes sense to talk of the tangent space and the normal space at a point. The tangent space is a linear/affine $d$-dimensional space and the normal space is a linear/affine $(n - d)$-dimensional space. Both pass through the point and are mutually orthogonal.
(2) For a function $z = f(x, y)$, the tangent plane to the graph of this function (a surface in $\mathbb{R}^3$) at the point $(x_0, y_0, f(x_0, y_0))$ *such that $f$ is differentiable at the point* $(x_0, y_0)$ is the plane:

$$z - f(x_0, y_0) = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

The corresponding linear function we get is:

$$L(x, y) = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

This provides a linear approximation to the function near the point where we are computing the tangent plane.

(3) It may be the case that a function $f$ of two variables is not differentiable at a point in its domain but the partial derivatives exist. In this case, although the *above formula makes sense as a formula*, the plane it gives *is not the tangent plane* – in fact, *no tangent plane exists*. Similarly, *no linearization exists*, and the linear function given by the above formula *is not a close approximation to the function near the point*.

3.5. **Chain rule.** Words ...

(1) The general formulation of chain rule: consider a function with $m$ inputs and $n$ outputs, and another function with $n$ inputs and $p$ outputs. Composing these, we get a function with $m$ inputs and $p$ outputs. The $m$ original inputs are termed *independent variables*, the $n$ in-between things are termed *intermediate variables*, and the $p$ final outputs are termed *dependent variables*.

For a given triple of independent variable $t$, intermediate variable $x$, and dependent variable $u$, the partial derivative of $u$ with respect to $t$ via $x$ is defined as:

$$\frac{\partial u}{\partial x} \frac{\partial x}{\partial t}$$

The chain rule says that the partial derivative of $u$ with respect to $t$ is the sum, over all intermediate variables, over the partial derivatives via each intermediate variable.

(2) The $1 \to 2 \to 1$ and $2 \to 2 \to 1$ versions (see the lecture notes or the book).

(3) There is also a tree interpretation of this, where we make pathways based on the directions/paths of dependence. This is discussed in the book, not the lecture notes.

(4) The product rule for scalar functions can be proved using the chain rule. Other variants of the product rule can be proved using generalized formulations of the chain rule, which are beyond our current scope.

(5) Implicit differentiation can be understood in terms of the chain rule and partial derivatives.

## 4. Double and iterated integrals

Words ...

(1) The double integral of a function $f$ of two variables, over a domain $D$ in $\mathbb{R}^2$, is denoted $\int\int_D f(x,y)\,dA$ and measures an infinite analogue of the sum of $f$-values at all points in $D$.

(2) Fubini's theorem for rectangles states that if $F$ is a function of two variables on a rectangle $R = [a,b] \times [p,q]$, such that $F$ is continuous except possibly at the union of finitely many smooth curves, then the integral equals either of these iterated integrals:

$$\int\int_R F(x,y)\,dA = \int_a^b \int_p^q F(x,y)\,dy\,dx = \int_p^q \int_a^b F(x,y)\,dx\,dy$$

(3) For a function $f$ defined on a closed connected bounded domain $D$ with a smooth boundary, we can make sense of $\int\int_D f(x,y)\,dA$ as being $\int\int_R F(x,y)\,dA$ where $R$ is a rectangular region containing $D$ and $F$ is a function that equals $f$ on $D$ and is 0 on the rest of $R$.

(4) Suppose $D$ is a Type I region, i.e., its intersection with every vertical line is either empty or a point or a line segment. Then, we can describe $D$ as $a \le x \le b$, $g_1(x) \le y \le g_2(x)$, where $g_1$ and $g_2$ are continuous functions. The integral $\int\int_D f(x,y)\,dA$ becomes:

$$\int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx$$

(5) Suppose $D$ is a Type II region, i.e., its intersection with every horizontal line is either empty or a point or a line segment. Then, we can describe $D$ as $p \le y \le q$, $g_1(y) \le x \le g_2(y)$, where $g_1$ and $g_2$ are continuous functions. The integral $\int\int_D f(x,y)\,dA$ becomes:

$$\int_p^q \int_{g_1(y)}^{g_2(y)} f(x,y)\,dx\,dy$$

(6) The double integral of $f+g$ over $D$ is the sum of the double integral of $f$ over $D$ and the double integral of $g$ over $D$. Similarly, scalars can be pulled out of double integrals.

(7) The integral of the function 1 over a domain is the area of the domain.
(8) If $f(x, y) \geq 0$ on a domain $D$, the integral of $f$ over $D$ is also $\geq 0$.
(9) If $f(x, y) \geq g(x, y)$ on a domain $D$, the integral of $f$ over $D$ is $\geq$ the integral of $g$ over $D$.
(10) If $m \leq f(x, y) \leq M$ over a domain $D$, then $\int\int_D f(x, y)\, dA$ is betweem $mA$ and $MA$ where $A$ is the area of $D$.
(11) If $f(x, y)$ is odd in $x$ and the domain of integration is symmetric about the $y$-axis, the integral is zero. If $f(x, y)$ is odd in $y$ and the domain is symmetric about the $x$-axis, the integral is zero.

Actions ...

(1) To compute a double integral, compute it as an iterated integral. For a rectangle, we can choose either order of integration, as long as the integration is feasible. For other types of regions, we need to first determine whether the region is Type I or Type II, and break it up into pieces of those types.
(2) For a multiplicatively separable function over a rectangular region (or for a sum of such multiplicatively separable functions), things are particularly easy.
(3) Sometimes, an integral cannot be computed using a particular order of integration – we might get stuck on the inner or the outer stage. However, it may be computable using the other order of integration.
(4) We can often use symmetry-based techniques to argue that certain parts of the integrand integrate to zero.
(5) Even in cases where the integral cannot be computed, we can bound it between limits using maximum or minimum values of function and/or using bigger or smaller regions on which the integral can be computed.