REVIEW SHEET FOR MIDTERM 2: ADVANCED

MATH 195, SECTION 59 (VIPUL NAIK)

To maximize efficiency, please bring a copy (print or readable electronic) of this review sheet to the review session.

The document does not include material that was part of the midterm 1 syllabus. Very little of that material will appear directly in midterm 2; however, you should have reasonable familiarity with the material.

1. Formula summary

1.1. Formula formulas.

(1) Unit vectors parallel to a nonzero vector v are $v/|v|$ and $-v/|v|$.

(2) Coordinates of the unit vector are the direction cosines. If $v/|v| = \langle \ell, m, n \rangle$, these are the direction cosines. If $\alpha, \beta, \gamma \in [0, \pi]$ are such that $\cos\alpha = \ell$, $\cos\beta = m$, $\cos\gamma = n$, then α, β, γ are the direction angles.

(3) Parametric equation of line in $\mathbb{R}^3$: $\mathbf{r} = \mathbf{r_0} + t\mathbf{v}$, $\mathbf{r_0}$ is the radial vector for a point in the line, $\mathbf{v}$ is the difference vector between two points in the line. In scalar terms, $x = x_0 + ta$, $y = y_0 + tb$, $z = z_0 + tc$, where $\mathbf{r} = \langle x, y, z \rangle$, $\mathbf{r_0} = \langle x_0, y_0, z_0 \rangle$, and $\mathbf{v} = \langle a, b, c \rangle$. (See also two-point form parametric equation).

(4) Symmetric equation of line in $\mathbb{R}^3$ not parallel to any coordinate plane (i.e., $abc \neq 0$ case):

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

with same notation as for parametric equation. (See also cases of parallel to coordinate plane).

(5) Equation of plane: vector equation $\mathbf{n} \cdot \mathbf{r} = \mathbf{n} \cdot \mathbf{r_0}$ where $\mathbf{n}$ is a nonzero normal vector, $\mathbf{r_0}$ is a fixed point in the plane. If $\mathbf{n} = \langle a, b, c \rangle$, $\mathbf{r_0} = \langle x_0, y_0, z_0 \rangle$, and $\mathbf{r} = \langle x, y, z \rangle$, we get:

$$ax + by + cz = ax_0 + by_0 + cz_0$$

(6) For a function $z = f(x, y)$, the tangent plane to the graph of this function (a surface in $\mathbb{R}^3$) at the point $(x_0, y_0, f(x_0, y_0))$ *such that f is differentiable at the point* (x_0, y_0) is the plane:

$$z - f(x_0, y_0) = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

The corresponding linear function we get is:

$$L(x, y) = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

This provides a linear approximation to the function near the point where we are computing the tangent plane.

1.2. Artistic formulas.

(1) Partial differentiation, multiplicatively separable – differentiate each piece in the corresponding variable the corresponding number of times.

(2) Partial differentiation, additively separable – pure partials, just care about function of that variable, mixed partials are zero.

(3) Integration along rectangular region, multiplicatively separable – product of integrals for function of each variable.

(4) Integration along non-rectangular region, multiplicatively separable – outer variable function can be pulled to outer integral.

2. Equations of lines and planes

2.1. Direction cosines. Error-spotting exercises ...

(1) If α, β, γ are the direction angles of the vector $\langle a, b, c\rangle$ then the direction angles of the vector $\langle -a, b, c\rangle$ are $-\alpha, \beta, \gamma$.

2.2. Lines. Error-spotting exercises ...

(1) *Counting issues*: They say that to describe a line in $\mathbb{R}^3$, we need $3 - 1 = 2$ equations in a top down description. However, the symmetric equation of a line:

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

is a *single* equation that describes the line.

(2) *Unparalleled lines*: By definition, if two lines do not intersect, they are parallel. Thus, the x-axis is parallel to the line $x = 1 + u$, $y = 2 + u$, $z = 3 + u$.

(3) *And and/or or*: Consider the planes $x + y + z = 0$ and $2x + 3y + 4z = 0$. Their intersection is a line given by the equation $(x + y + z)(2x + 3y + 4z) = 0$.

2.3. Planes. No error-spotting exercises.

3. Functions of several variables

3.1. Introduction. Error-spotting exercises ...

(1) *One-point curves*: Consider the function $f(x, y) := (x - 1)^2 + (y + 1)^2 - 3$. The level "curve" for the value -3 is the single point $(1, -1)$. This is a point, not a curve at all. So, the claim that level curves are one-dimensional is wrong, and the term "curve" itself is a misnomer.

(2) *Count issues again*: Consider the function $f(x, y) := x^2 - y^2$. The level "curve" for the value 1 is a union of two curves, one on the positive x-axis side and the other on the negative x-axis side. The level curve thus isn't a curve at all, it is a union of multiple curves.

(3) *A new unparalleled level*: Consider the function $f(x, y, z) := ax + by + cz$ of three variables. The level curves of this function are the lines parallel to the vector $\langle a, b, c\rangle$.

3.2. Limits and continuity. Error-spotting exercises ...

(1) *Zero ain't infinity*: Consider the limit $\lim_{(x,y)\to(0,0)} (x^4 + y^4)/(x^2 + y^2)^2$. We see that the numerator and denominator are both homogeneous polynomials of degree four, and so the limit of the quotient is the quotient of the leading coefficients, which are both 1. So the limit of the quotient is 1. We can verify this by noting that the limit for approach along the x-axis as well as the y-axis are both equal to 1.

(2) *Curvophobia* or *straightonormativity*: To verify that the limit of a function at the origin equals a particular value, we need to compute the limit along the x-axis, along the y-axis, and along the line $y = mx$ for m fixed but arbitrary. If all the three answers are a constant independent of m, then that is the limit.

3.3. Partial derivatives. Error-spotting exercises ...

(1) *Once it's fixed, it stays fixed*: Here is a simple logical explanation as to why, for any function f of two variables x and y, the second-order mixed partial derivative f_{xy} must be zero. Recall that f_x is the first-order partial derivative of x holding y constant. In other words, we fix the value of y and are allowed to vary only x, and measure the rate of change of f subject to that restriciton.

The second-order mixed partial derivative $f_{xy} = (f_x)_y$ is obtained by taking the first-order partial f_x and figuring out how it changes with respect to y holding x constant. But note from the preceding paragraph that y needs to be held constant in order to make sense of f_x. Thus, for computing f_{xy}, *both* x and y need to be held constant. Since both coordinates are being held constant, there is no scope for f to change, hence f_{xy} is zero.

(2) *Mixed up partials*: To differentiate a multiplicatively separable function, we differentiate the function of x with respect to x the required number of times and the function of y with respect to y the required number of times, and then multiply. Thus, if $f(x, y) := \sin(x^2 \sin y)$, we get $f_{xy}(x, y) = \cos(2x \cos y)$.

(3) *Slaving for joy*: My happiness is proportional to the logarithm of my income; every time my income doubles, my happiness goes up 0.3 units. I have observed that my income obeys increasing returns to effort, and empirically I find that my total income is proportional to the $(4/3)^{th}$ power of the number of hours I work. Therefore, my happiness also obeys increasing returns to effort.

(4) *Futility personified*: Consider a production function $f(L, K) = (\min\{L, K\})^2$. We know that if $L > K$, then $f_L(L, K) = 0$. This means that reducing the value of L has no impact on the output. But if that's true, then L can be reduced to 0, and output would be unaffected. Similarly, K can be reduced to zero, and output would be unaffected. But that's nonsense.

(5) *Mixed up partials – something doesn't add up*: Suppose $F(x, y) := f(x) + g(y)$. Then $F_{xy}(x, y) = f'(x) + g'(y)$.

(6) *Mixed up partials – shut up and multiply*: Suppose $F(x, y) := f(x)g(y)$. We know that the mixed partial $F_{xy}(x, y) = f'(x)g'(y)$. But this is in contradiction with the product rule, which states that the derivative of the product is *not* the product of the derivatives. Shouldn't the answer be $f'(x)g(y) + f(x)g'(y)$?

(7) *Quid est quod custodire cupis constans*: Let f be a function of two variables. Define $g(x, y) := f(x, x + y)$. Then, clearly, $g(2, 3) = f(2, 5)$. Hence also, we have $g_1(2, 3) = f_1(2, 5)$ (where the subscript $_1$ denotes partial differentiation with respect to the first input keeping the second input constant).

(8) *Value depends only on the variable you differentiate with respect to*: A manager wants to figure out the marginal product of labor. He has an expression for the production function in terms of labor and capital. In order to calculate the marginal product of labor, he simply needs to know the current labor expenditure to plug into the formula. Information on the current capital expenditures is redundant.

3.4. **Tangent planes and linear approximations.** Error-spotting exercises...

(1) *The rational elite and the irrational hoi polloi are on different planes*: Consider the function:

$$f(x, y) := \begin{cases} 1, & x \text{ rational or } y \text{ rational} \\ 0, & x \text{ and } y \text{ both irrational} \end{cases}$$

Suppose x_0, y_0 are rational numbers, so (x_0, y_0) is a point both of whose coordinates are rational. Then, we have $f(x_0, y_0) = 1$ and $f_x(x_0, y_0) = f_y(x_0, y_0) = 0$. Thus, we get that the tangent plane to the graph of f through the point $(x_0, y_0, f(x_0, y_0))$ is:

$$z = 1 + 0(x - x_0) + 0(x - x_0)$$

So we get that the equation is:

$$z = 1$$

(2) *So near, yet so far, or, missing the forest for the trees, or, going off on tangents*: The tangent line to $(0, 0)$ for the curve $y = \sin x$ in the xy-plane is the $y = x$ line. This is therefore the best straight line approximation to the curve. Thus, for instance, a reasonable approximation for sin(1000) is 1000.

3.5. **Chain rule.** Error-spotting exercises ...

(1) *x, tx, it's all the same*: Suppose $f(x, y)$ is a function of two variables. Then, we have:

$$f_x(tx, ty) = \frac{\partial}{\partial x}[f(tx, ty)]$$

Note: The underlying issue here affected some people's attempts at advanced HW 6 question 5.

(2) *Functions are born free, yet everywhere they are in chains*: Suppose f and g are functions of one variable. Then, we know that:

$$(f \circ g)'(t) = f'(g(t))g'(t)$$

by the chain rule. Differentiating both sides with respect to t again, and using the product rule, we get:

$$(f \circ g)''(t) = \frac{d}{dt}\left[f'(g(t))g'(t) + f'(g(t))g''(t)\right] = f''(g(t))g'(t) + f'(g(t))g''(t)$$

(3) *On the other hand*: Suppose $z = f(x, y)$ where $x = g(t)$ and $y = h(t)$. Then, we have:

$$\frac{\partial f_x}{\partial t} = \frac{\partial f_x}{\partial x}\frac{\partial x}{\partial t}$$

4. Double and iterated integrals

Error-spotting exercises ...

(1) *Fundamental theorem of miscalculus*: Suppose we are integrating a continuous function $g(x, y)$ of two variables over a rectangular region $[a, b] \times [p, q]$. Then, if $G_{xy} = g$, the value of the integral is $G(b, q) - G(a, p)$. This is just like the fundamental theorem of calculus.

(2) *Separation of abscissa and ordinate*: Suppose $F(x, y) := f(x)g(y)$. We want to integrate F on the region $0 \le x \le 5$, $0 \le y \le x^2$. Since F is multiplicatively separable, we don't need to compute this as an iterated integral, and instead, we can compute it as a product:

$$\left(\int_0^5 f(x)\, dx\right)\left(\int_0^{x^2} g(y)\, dy\right)$$

(3) *Dissolving the bonds of addition*: Suppose $F(x, y) := f(x) + g(y)$, and we need to integrate F on $[a, b] \times [p, q]$. The integral is:

$$\int_a^b f(x)\, dx + \int_p^q g(y)\, dy$$

(4) *Argument from personal incredulity*: The double integral for a function F on a domain D exists only if D is a Type I or Type II region.

(5) *Another argument from personal incredulity*: e^{-x^2} is not an integrable function of one variable, i.e., it does not have an antiderivative.

(6) *Straightonormativity yet again*: If $F(x, y) = f(x)g(y)$ and we have antiderivatives available for f and g, we can use these to successfully integrate F over any closed bounded convex region.

(7) *O mirror to my soul, don't be orthogonal!*: If f is a function and D is a closed convex region centered at the origin symmetric about the x-axis, such that f is odd in x for each fixed value of y, then the integral of f over D is zero.

(8) *Positivity bias yet again, or tunnel vision*: The integral:

$$\int_2^3 \frac{dx}{x^2 + y}$$

gives us:

$$\left[\frac{1}{\sqrt{y}} \arctan\left(\frac{x}{\sqrt{y}}\right)\right]_{x=2}^{x=3}$$

This simplifies to:

$$\frac{1}{\sqrt{y}}\left[\arctan\left(\frac{3}{\sqrt{y}}\right) - \arctan\left(\frac{2}{\sqrt{y}}\right)\right]$$

(9) Consider the following integral on the region $D = [0, a] \times [0, a]$ for the function $f(x, y) := g[(\max\{x, y\})^2]$. We get:

$$\int\int_D f(x, y)\, dA = \max\{\int_0^a g(x^2)\, dx, \int_0^a g(y^2)\, dy\}$$

Since both integrals are the same, this becomes:

$$\int\int_D f(x,y)\,dA = \int_0^a g(x^2)\,dx$$

If G is an antiderivative for g, this becomes:

$$[G(x^2)]_0^a$$

This simplifies to $G(a^2) - G(0)$.