TAKE-HOME CLASS QUIZ SOLUTIONS: WEDNESDAY JANUARY 9: PARAMETRIC STUFF

MATH 195, SECTION 59 (VIPUL NAIK)

1. Performance review

26 people took this 11-question quiz. The score distribution was as follows:

- Score of 4: 1 person.
- Score of 6: 3 people.
- Score of 7: 7 people.
- Score of 8: 1 person.
- Score of 9: 6 people.
- Score of 10: 4 people.
- Score of 11: 4 people.

The question-wise answers and performance review are below.

(1) Option (E): 19 people.
(2) Option (C): 13 people.
(3) Option (C): 25 people.
(4) Option (A): 24 people.
(5) Option (A): 20 people.
(6) Option (D): 25 people.
(7) Option (A): 25 people.
(8) Option (C): 20 people.
(9) Option (B): 11 people.
(10) Option (D): 12 people.
(11) Option (E): 23 people.

2. Solutions

(1) Consider the curve given by the parametric description $x = \cos t$, $y = \sin t$, where t varies over the interval $[a, b]$ with $a < b$. What is a necessary and sufficient condition on a and b for this curve to be the circle $x^2 + y^2 = 1$?
(A) $b - a = \pi$
(B) $b - a > \pi$
(C) $b - a = 2\pi$
(D) $b - a > 2\pi$
(E) $b - a \geq 2\pi$

Answer: Option (E)

Explanation: The curve is traced along the circle starting at $(\cos a, \sin a)$ and going around the circle till we reach b. In order to cover the whole circle, it is necessary that it make at least one full angle of 2π. Thus, the condition $b - a \geq 2\pi$.

Note that the equality case is valid because we are working with the *closed* interval $[a, b]$. If we were working with the open interval (a, b), then strict inequality would be the necessary and sufficient condition.

Performance review: 19 out of 26 got this. 7 chose C, which is the correct answer for the curve to *just* cover the circle but is the wrong choice for a necessary and sufficient condition.

Historical note (last time): 11 people got this correct. 9 people chose (C), 2 people chose (B), 1 person each chose (A) and (D).

(2) (*) Consider the curve given by the parametric description $x = \arctan t$ and $y = \arctan t$ for $t \in \mathbb{R}$. Which of the following is the best description of this curve?
(A) It is the graph of the function arctan
(B) It is the line $y = x$
(C) It is a line segment (without endpoints) that is part of the line $y = x$
(D) It is a half-line (with endpoint) that is part of the line $y = x$
(E) It is a disjoint union of two half-lines that are both part of the line $y = x$
Answer: Option (C)
Explanation: Eliminating the parameter t, we get that $y = x$, but with the additional caveat that the value of x (hence also y) must be in the range of arctan. The range of arctan is the open interval $(-\pi/2, \pi/2)$, thus we get the corresponding line segment without endpoints joining the point with coordinates $(\pi/2, \pi/2)$ to the point with coordinates $(-\pi/2, -\pi/2)$.
Performance review: 13 out of 26 got this. 10 chose (D), 2 chose (B), 1 chose multiple options.
Historical note (last time): 8 people got this correct. 9 people chose (B), which would be the right idea *except for the issue of domain/range restrictions*. 4 chose (D), 2 chose (A), 1 chose (E).

(3) (*) Consider the curve given by the parametric description $x = \sin^2 t$ and $y = \cos^2 t$ for $t \in \mathbb{R}$. Which of the following is the best description of this curve?
(A) It is the arc of the circle $x^2 + y^2 = 1$ comprising the first quadrant, i.e., when $x \geq 0$ and $y \geq 0$.
(B) It is the entire circle $x^2 + y^2 = 1$
(C) It is the line segment joining the points $(0, 1)$ and $(1, 0)$
(D) It is the line $y = 1 - x$
(E) It is a portion of the parabola $y = x^2$
Answer: Option (C)
Explanation: Eliminating the parameter, we obtain that $x + y = 1$. Further, we much have $x \geq 0$ and $y \geq 0$ since they are both squares. Subject to these conditions, any pair (x, y) works. This is thus the part of the line $x + y = 1$ which lies in the first quadrant. This can alternatively be described as the line segment joining the points $(0, 1)$ and $(1, 0)$.
Performance review: 25 out of 26 got this. 1 chose (B).
Historical note (last time): 5 people got this correct. 11 people chose (D), which would be the correct answer *except for the issue of domain/range restrictions*. 4 people each chose (A) and (B).

(4) Identify the parametric description which *does not* correspond to the set of points (x, y) satisfying $x^3 = y^5$.
(A) $x = t^3$, $y = t^5$, for $t \in \mathbb{R}$
(B) $x = t^5$, $y = t^3$, for $t \in \mathbb{R}$
(C) $x = t$, $y = t^{3/5}$, for $t \in \mathbb{R}$
(D) $x = t^{5/3}$, $y = t$, for $t \in \mathbb{R}$
(E) All of the above parametric descriptions work
Answer: Option (A)
Explanation: The exponents are at the wrong places – if $x = t^3$, then $x^3 = t^9$ and if $y = t^5$, then $y^5 = t^{25}$ – these are certainly not equal.
Performance review: 24 out of 26 got this. 2 chose (D).
Historical note (last time): 16 people got this correct. 4 chose (E), 3 chose (B), 1 chose (C).

(5) (*) Consider the parametric description $x = f(t)$, $y = g(t)$ where t varies over all of $\mathbb{R}$. What is the necessary and sufficient condition for the curve given by this to be the graph of a function, i.e., to satisfy the vertical line test?
(A) For any t_1 and t_2 satisfying $f(t_1) = f(t_2)$, we must have $g(t_1) = g(t_2)$.
(B) For any t_1 and t_2 satisfying $g(t_1) = g(t_2)$, we must have $f(t_1) = f(t_2)$.
(C) Both f and g are one-to-one functions.
(D) For any t_1 and t_2, we must have $f(t_1) = f(t_2)$.
(E) For any t_1 and t_2, we must have $g(t_1) = g(t_2)$.
Answer: Option (A)

Explanation: We want that for a given x-value there is at most one y-value (the vertical line test). This means that if, at two times t_1 and t_2, the x-values $f(t_1)$ and $f(t_2)$ are equal to each other, the y-values $g(t_1)$ and $g(t_2)$ must also be equal to each other. This is option (A).

Performance review: 20 out of 26 got this. 3 each chose (B) and (C).

Historical note (last time): 10 people got this correct. 11 chose (C), which is a *sufficient* but not a necessary condition. 2 chose (B) and 1 chose (D).

(6) Suppose f and g are both twice differentiable functions everywhere on $\mathbb{R}$. Which of the following is the correct formula for $(f \circ g)''$?
(A) $(f'' \circ g) \cdot g''$
(B) $(f'' \circ g) \cdot (f' \circ g') \cdot g''$
(C) $(f'' \circ g) \cdot (f' \circ g') \cdot (f \circ g'')$
(D) $(f'' \circ g) \cdot (g')^2 + (f' \circ g) \cdot g''$
(E) $(f' \circ g') \cdot (f \circ g) + (f'' \circ g'')$

Answer: Option (D)

Explanation: This question is tricky because it requires the application of both the product rule and the chain rule, with the latter being used twice. We first note that:

$$(f \circ g)' = (f' \circ g) \cdot g'$$

Now, we differentiate both sides:

$$(f \circ g)'' = [(f' \circ g) \cdot g']'$$

The expression on the right side that needs to be differentiated is a product, so we use the product rule:

$$(f \circ g)'' = [(f' \circ g)' \cdot g'] + [(f' \circ g) \cdot g'']$$

Now, the inner composition $f' \circ g$ needs to be differentiated. We use the chain rule and obtain that $(f' \circ g)' = (f'' \circ g) \cdot g'$. Plugging this back in, we get:

$$(f \circ g)'' = (f'' \circ g) \cdot (g')^2 + (f' \circ g) \cdot g''$$

Remark: What's worth noting here is that in order to differentiate composites of functions, you need to use both composites *and* products (that's the chain rule). And in order to differentiate products, you need to use both products *and* sums (that's the product rule). Thus, in order to differentiate a composite twice, we need to use composites, products, *and* sums.

Performance review: 25 out of 26 got this. 1 chose (A).

Historical note (last time): 20 out of 21 people got this correct. 1 person left the question blank.

Historical note: I put this question in a quiz for Math 152 back in October 2010, and 14 of 14 people who took that quiz got it correct.

(7) Suppose $x = f(t)$ and $y = g(t)$ where f and g are both twice differentiable functions. What is d^2y/dx^2 in terms of f and g and their derivatives evaluated at t?
(A) $(f'(t)g''(t) - g'(t)f''(t))/(f'(t))^3$
(B) $(f'(t)g''(t) - g'(t)f''(t))/(g'(t))^3$
(C) $(g'(t)f''(t) - f'(t)g''(t))/(f'(t))^3$
(D) $(g'(t)f''(t) - f'(t)g''(t))/(g'(t))^3$
(E) None of the above

Answer: Option (A)

Explanation: See lecture notes.

Performance review: 25 out of 26 got this. 1 chose (B).

Historical note (last time): 20 out of 21 people got this correct. 1 person chose (E).

(8) Which of the following pair of bounds works for the arc length for the portion of the graph of the sine function between $(a, \sin a)$ and $(b, \sin b)$ where $a < b$?
(A) Between $(b - a)/\sqrt{3}$ and $(b - a)/\sqrt{2}$
(B) Between $(b - a)/\sqrt{2}$ and $b - a$

(C) Between $(b-a)$ and $\sqrt{2}(b-a)$
(D) Between $\sqrt{2}(b-a)$ and $\sqrt{3}(b-a)$
(E) Between $\sqrt{3}(b-a)$ and $2(b-a)$
Answer: Option (C)
Explanation: The derivative function is cos, so the corresponding arc length formula gives:

$$\int_a^b \sqrt{1+\cos^2 x}\,dx$$

The integrand is always between 1 and $\sqrt{2}$, so the integral must be between $1 \cdot (b-a)$ and $\sqrt{2} \cdot (b-a)$.

Performance review: 20 out of 26 got this. 4 chose (D), 1 each chose (A) and (B).

Historical note (last time): 15 out of 21 people got this correct. 2 chose (B), 2 left the question blank, 1 each chose (A) and (D).

(9) (*) Consider the parametric curve $x = e^t$, $y = e^{t^2}$. How does y grow in terms of x as $x \to \infty$?
(A) y grows like a polynomial in x.
(B) y grows faster than any polynomial in x but slower than an exponential function of x.
(C) y grows exponentially in x.
(D) y grows super-exponentially in x but slower than a double exponential in x.
(E) y grows like a double exponential in x.
Answer: Option (B)
Explanation: Note that a polynomial in x is still exponential in t, and not in t^2, i.e., it is too slow to be y. Thus y grows faster than any polynomial in x. On the other hand, an exponential in x is doubly exponential in t, which is faster in growth than e^{t^2}. Thus, option (B).

Performance review: 11 out of 26 got this. 9 chose (D), 3 chose (E), 2 chose (C), 1 chose (A).

Historical note (last time): 7 out of 21 people got this correct. 4 each chose (A) and (E), 3 chose (C), 1 chose (D), and 2 left the question blank.

(10) (*) We say that a curve is *algebraic* if it admits a parameterization of the form $x = f(t)$, $y = g(t)$, where f and g are rational functions and t varies over some subset of the real numbers. Which of the following curves is *not* algebraic?
(A) $x = \cos t$, $y = \sin t$, $t \in \mathbb{R}$
(B) $x = \cos t$, $y = \cos(3t)$, $t \in \mathbb{R}$
(C) $x = \cos t$, $y = \cos^2 t$, $t \in \mathbb{R}$
(D) $x = \cos t$, $y = \cos(t^2)$, $t \in \mathbb{R}$
(E) None of the above, i.e., they are all algebraic
Answer: Option (D)
Explanation: In all the other cases, we can elucidate an algebraic relationship between the variables.

For option (A), we can set both $\cos t$ and $\sin t$ as rational functions in $\tan(t/2)$. In fact, the rational functions in $\tan(t/2)$ approach works for options (B) and (C) as well, though there are simpler approaches in those cases. The approach does not work for option (D).

Performance review: 12 out of 26 got this. 9 chose (E), 3 chose (A), 1 each chose (B) and (C).

Historical note (last time): 11 out of 21 people got this correct. 8 chose (E), 1 chose (B), 1 left the question blank.

(11) (+) Suppose $x = f(t)$, $y = g(t)$, $t \in \mathbb{R}$ is a parametric description of a curve Γ and both f and g are continuous on all of $\mathbb{R}$. If both f and g are even, what can we conclude about Γ and its parameterization?
(A) Γ is symmetric about the y-axis
(B) Γ is symmetric about the x-axis
(C) Γ is symmetric about the line $y = x$
(D) Γ has half turn symmetry about the origin
(E) The parameterizations of Γ for $t \leq 0$ and for $t \geq 0$ both cover all of Γ, and in directions mutually reverse to each other.

Answer: Option (E)

Explanation: See lecture notes.

Performance review: 23 out of 26 got this. 2 chose (A), 1 chose (C).

Historical note (last time): 5 out of 21 people got this correct. 7 chose (A), 4 chose (D), 2 each chose (B) and (C), 1 left the question blank.

CLASS QUIZ SOLUTIONS: FRIDAY JANUARY 11: POLAR COORDINATES

MATH 195, SECTION 59 (VIPUL NAIK)

1. Performance review

27 people took this 3-question quiz. The score distribution was as follows:

- Score of 1: 1 person
- Score of 2: 8 people
- Score of 3: 18 people

The answers and performance review for individual questions are:

(1) Option (B): 26 people.
(2) Option (D): 22 people.
(3) Option (A): 23 people.

2. Solutions

(1) Consider a straight line that does not pass through the pole in a polar coordinate system. The equation of such a line in the polar coordinate system can be expressed as $r = F(\theta)$. What kind of function is F?
 (A) $F(\theta)$ is a linear combination of $\sin\theta$ and $\cos\theta$
 (B) $F(\theta)$ is the reciprocal of a linear combination of $\sin\theta$ and $\cos\theta$.
 (C) $F(\theta)$ is a linear combination of $\tan\theta$ and $\cot\theta$.
 (D) $F(\theta)$ is the reciprocal of a linear combination of $\tan\theta$ and $\cot\theta$.
 (E) $F(\theta)$ is a linear combination of $\sec\theta$ and $\csc\theta$.

 Answer: Option (B)

 Explanation: Consider the corresponding Cartesian coordinate system. The Cartesian equation of a straight line is of the form $ax + by = c$. Since the line does not pass through the origin/pole, $c \neq 0$. Set $x = r\cos\theta$ and $y = r\sin\theta$, and we get $r(a\cos\theta + b\sin\theta) = c$. Rearranging, we obtain that $r = c/(a\cos\theta + b\sin\theta)$. We could rewrite as $r = 1/((a/c)\cos\theta + (b/c)\sin\theta)$.

 Performance review: 26 out of 27 got this. 1 chose (C).

 Historical note (last time): 8 out of 21 people got this correct. 6 chose (C), 5 chose (A), 1 each chose (D) and (E).

(2) Consider the curve $r = \sin^2\theta$. Which of the following symmetries does the curve enjoy? Please see Options (D) and (E) before answering.
 (A) Mirror symmetry about the polar axis
 (B) Mirror symmetry about an axis perpendicular to the polar axis (what would be the y-axis if the polar axis is the x-axis)
 (C) Half turn symmetry about the pole
 (D) All of the above
 (E) None of the above

 Answer: Option (D)

 Explanation: We use the fact that $\sin^2\theta = \sin^2(-\theta)$ to deduce mirror symmetry about the polar axis. We use that $\sin^2\theta = \sin^2(\pi - \theta)$ to deduce mirror symmetry about the y-axis. Finally, we use that $\sin^2\theta = \sin^2(\pi + \theta)$ to deduce half turn symmetry about the pole.

 Performance review: 22 out of 27 got this. 5 chose (B).

 Historical note (last time): 10 out of 21 people got this correct. 6 chose (B), 5 chose (A).

(3) Which of the following is the correct expression for the length of the part of the curve $r = F(\theta)$ from $\theta = \alpha$ to $\theta = \beta$, with $\alpha < \beta$?

(A) $\int_\alpha^\beta \sqrt{(F(\theta))^2 + (F'(\theta))^2}\, d\theta$
(B) $\int_\alpha^\beta |F(\theta) + F'(\theta)|\, d\theta$
(C) $\int_\alpha^\beta |F(\theta) - F'(\theta)|\, d\theta$
(D) $\int_\alpha^\beta \sqrt{(F(\theta))^2 + (F'(\theta))^2 + 4F(\theta)F'(\theta)}\, d\theta$
(E) $\int_\alpha^\beta \sqrt{(F(\theta))^2 + (F'(\theta))^2 - 4F(\theta)F'(\theta)}\, d\theta$

Answer: Option (A)

Explanation: There are many ways of seeing this, including a direct justification in polar coordinates, but we provide an easy explanation using the Cartesian coordinates. We know that:

$$x = F(\theta)\cos\theta, \qquad y = F(\theta)\sin\theta$$

We get:

$$\frac{dx}{d\theta} = F'(\theta)\cos\theta - F(\theta)\sin\theta, \qquad \frac{dy}{d\theta} = F'(\theta)\sin\theta + F(\theta)\cos\theta$$

Squaring and adding, we know that the $2F(\theta)F'(\theta)\cos\theta\sin\theta$ term cancels between the two expressions, and we are left with:

$$\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 = (F'(\theta))^2\cos^2\theta + (F(\theta))^2\sin^2\theta + (F'(\theta))^2\sin^2\theta + (F(\theta))^2\cos^2\theta = [(F'(\theta))^2 + (F(\theta))^2](\cos^2\theta + \sin^2\theta)$$

Using $\cos^2\theta + \sin^2\theta = 1$, we get the desired result.

Performance review: 23 out of 27 got this. 2 chose (D), 1 each chose (B) and (E).

Historical note (last time): 14 out of 21 people got this correct. 4 chose (D), 3 chose (E).

TAKE-HOME CLASS QUIZ SOLUTIONS: DUE WEDNESDAY JANUARY 16: THREE DIMENSIONS

MATH 195, SECTION 59 (VIPUL NAIK)

1. Performance review

25 people took this quiz. The score distribution is as follows:

- Score of 2: 2 people
- Score of 3: 2 people
- Score of 4: 2 people
- Score of 5: 6 people
- Score of 6: 11 people
- Score of 7: 2 people

The mean score was about 5.12.

Here are the question-wise answers and performance summary (more details in the next section):

(1) Option (C): 16 people
(2) Option (B): 20 people
(3) Option (A): 19 people
(4) Option (C): 20 people
(5) Option (C): 24 people
(6) Option (A): 5 people
(7) Option (C): 24 people

2. Solutions

(1) (*) Consider the subset of $\mathbb{R}^3$ given by the condition $(x^2 + y^2 - 1)(y^2 + z^2 - 1)(x^2 + z^2 - 1) = 0$. What kind of subset is this?
   (A) It is a sphere centered at the origin and of radius 1.
   (B) It is the union of three circles, each centered at the origin and of radius 1, and lying in the xy-plane, yz-plane, and xz-plane respectively.
   (C) It is the union of three cylinders, each of radius 1, about the x-axis, y-axis, and z-axis respectively.
   (D) It is the intersection of three circles, each centered at the origin and of radius 1, and lying in the xy-plane, yz-plane, and xz-plane respectively.
   (E) It is the intersection of three cylinders, each of radius 1, about the x-axis, y-axis, and z-axis respectively.

   Answer: Option (C)

   Explanation: Each of the individual conditions gives a cylinder about one of the coordinate axes of radius 1. For instance, $x^2 + y^2 - 1 = 0$ gives the cylinder of radius 1 about the z-axis. For the product to be zero, one or more of the conditions must be satisfied, so we get the union.

   Performance review: 16 out of 25 got this. 7 chose (B), 2 chose (D).

   Historical note (last time): 12 out of 24 people got this correct. 3 chose (B), 4 chose (D), 3 chose (E), 2 chose (A).

(2) Given two distinct points A and B in three-dimensional space, what is the nature of the set of possibilities for a third point C such that AC and BC have equal length (i.e., C is equidistant from A and B)? *Didn't appear last time*
   (A) Sphere
   (B) Plane

(C) Circle
(D) Line
(E) Two points

Answer: Option (B)

Explanation: This is the plane perpendicular to the line segment AB and intersecting the line segment at its midpoint. It is the analogue in three dimensions of the perpendicular bisector in two dimensions.

Performance review: 20 out of 25 got this. 3 chose (A), 2 chose (C).

(3) Given two distinct points A and B in three-dimensional space, what is the nature of the set of possibilities for a third point C such that the triangle ABC is a right triangle with AB as its hypotenuse?
(A) Sphere (minus two points)
(B) Plane
(C) Circle (minus two points)
(D) Line
(E) Square

Answer: Option (A)

Explanation: By some elementary geometry, we know that this is the sphere with diameter AB. However, the points A and B themselves need to be excluded.

Note that if we were in a plane, we would get merely the circle with diameter AB minus two points. This seems to have been the most popular incorrect option chosen.

—em Performance review: 19 out of 25 got this. 3 chose (C), 2 chose (E), 1 chose (D).

Historical note (last time): 15 out of 24 people got this correct. 7 chose (C), 1 each chose (B) and (E).

(4) Given two distinct points A and B in three-dimensional space, what is the nature of the set of possibilities for a third point C such that the triangle ABC is a right isosceles triangle with AB as its hypotenuse? *Didn't appear last time.*
(A) Sphere
(B) Plane
(C) Circle
(D) Line
(E) Square

Answer: Option (C)

Explanation: It arises as the intersection of spheres centered at A and B of radius equal to $|AB|/\sqrt{2}$.

Performance review: 20 out of 25 got this. 4 chose (E), 1 chose (B).

(5) Given two distinct points A and B in three-dimensional space, what is the nature of the set of possibilities for a third point C such that the triangle ABC is equilateral?
(A) Sphere
(B) Plane
(C) Circle
(D) Line
(E) Two points

Answer: Option (C)

Explanation: It arises as an intersection of spheres centered at A and B with radius equal to AB. Alternatively, pick any one choice of C. The set of all choices can be obtained by revolving this point about the line of AB, and we get a circle.

Performance review: 24 out of 25 got this. 1 chose (D).

Historical note (last time): 23 out of 24 people got this correct (way to go, folks!). 1 person chose (B).

(6) Given two distinct points A and B in three-dimensional space, what is the nature of the set of possibilities for a third point C such that $|AC|/|BC| = \lambda$ for λ a fixed positive real number not equal to 1? *Didn't appear last time.*

(A) Sphere
(B) Plane
(C) Circle
(D) Line
(E) Square
Answer: Option (A)
Explanation: Use distance formula, simplify. Similar questions appear on your homework.
Performance review: 5 out of 25 got this. 7 chose (C), 6 each chose (B) and (D), 1 chose (E).

(7) Consider the parametric curve in three dimensions given by the coordinate description $t \mapsto (\cos t, \sin t, \cos(2t))$, with $t \in \mathbb{R}$. We can consider the *projections* of this curve onto the xy-plane, yz-plane, and xz-plane, which are basically what we get by dropping perpendiculars from the curve to these planes. What is the correct description of the curves obtained by doing the three projections?
(A) The projections on the xy-plane and yz-plane are both parts of parabolas, and the projection on the xz-plane is a circle.
(B) The projections on the xy-plane and yz-plane are both circles, and the projection on the xz-plane is a part of a parabola.
(C) The projection on the xy-plane is a circle, and the projections on the yz-plane and xz-plane are both parts of parabolas.
(D) The projection on the xy-plane is a part of a parabola, the projection on the xz-plane and yz-plane are both circles.
(E) All the three projections are circles.
Answer: Option (C)
Explanation: The projection on the xy-plane is just $t \mapsto (\cos t, \sin t)$, which is the unit circle. The projection on the xz-plane is $t \mapsto (\cos t, \cos(2t))$ and we have the quadratic relation $\cos(2t) = 2(\cos t)^2 - 1$, subject to domain restriction $\cos t \in [-1, 1]$. Thus, we get a part of a parabola. The projection on the yz-plane is $t \mapsto (\sin t, \cos(2t))$ and we have the quadratic relation $\cos(2t) = 1 - 2(\sin t)^2$, subject to domain restriction $\sin t \in [-1, 1]$. Thus, we get a part of a parabola.
Performance review: 24 out of 25 got this. 1 chose (A).
Historical note (last time): 17 out of 24 people got this correct. 4 chose (B) and 3 chose (D).

CLASS QUIZ SOLUTIONS: FRIDAY JANUARY 18: VECTORS

MATH 195, SECTION 59 (VIPUL NAIK)

1. Performance review

? people took this 6-question quiz. The score distribution was as follows:

- Score of 3: 3 people.
- Score of 4: 3 people.
- Score of 5: 17 people.
- Score of 6: 1 person.

The mean score was about 4.5. The question wise solutions and performance summary are below:

(1) Option (C): 21 people.
(2) Option (E): 23 people.
(3) Option (C): 20 people.
(4) Option (E): 21 people.
(5) Option (D): 3 people.
(6) Option (C): 24 people.

2. Solutions

(1) Suppose S is a collection of *nonzero* vectors in $\mathbb{R}^3$ with the property that the dot product of any two distinct elements of S is zero. What is the maximum possible size of S?
 (A) 1
 (B) 2
 (C) 3
 (D) 4
 (E) There is no finite bound on the size of S

 Answer: Option (C)

 Explanation: The given property indicates that any two elements of S are mutually orthogonal vectors. In $\mathbb{R}^3$, there can be at most three mutually orthogonal directions, because the space is three-dimensional. (This can be proved formally, but we won't bother here).

 Performance review: 21 out of 24 people got this. 3 chose (B).

 Historical note (last year): 14 out of 23 people got this correct. 6 chose (E), 2 chose (B), 1 chose (D).

(2) Suppose S is a collection of *nonzero* vectors in $\mathbb{R}^3$ such that the cross product of any two distinct elements of S is the zero vector. What is the maximum possible size of S?
 (A) 1
 (B) 2
 (C) 3
 (D) 4
 (E) There is no finite bound on the size of S

 Answer: Option (E)

 Explanation: The given condition means that any two elements of S are scalar multiples of each other. We can easily construct an infinite set with this property: consider *all* nonzero scalar multiples of a fixed nonzero vector. There are infinitely many such multiples, because there are infinitely many nonzero reals, and each such multiple is different.

 Performance review: 23 out of 24 people got this. 1 person chose (B).

Historical note (last year): 17 out of 23 people got this correct. 3 chose (A), 2 chose (B), 1 chose (C).

(3) Suppose a and b are vectors in $\mathbb{R}^3$. Which of the following is/are true?
(A) If both a and b are nonzero vectors, then $a \times b$ is a nonzero vector.
(B) If $a \times b$ is a nonzero vector, then $a \cdot (a \times b)$ is a nonzero real number.
(C) If $a \times b$ is a nonzero vector, then $a \times (a \times b)$ is a nonzero vector.
(D) All of the above
(E) None of the above

Answer: Option (C)

Explanation: If $a \times b$ is a nonzero vector, then it is in particular orthogonal to both a and b. Further, it also means that neither a nor b is zero. Thus, a and $a \times b$ are mutually orthogonal nonzero vectors, and in particular are not scalar multiples of each other. Thus, the cross product $a \times (a \times b)$ is nonzero.

Performance review: 20 out of 24 people got this. 2 each chose (D) and (E).

Historical note (last year): 6 out of 23 people got this correct. 11 chose (E), 4 chose (D), 1 chose (B), 1 left the question blank.

(4) (*) Suppose a, b, c, and d are vectors in $\mathbb{R}^3$, with $a \times b \neq 0$ and $c \times d \neq 0$. What does $(a \times b) \times (c \times d) = 0$ mean?
(A) Both the vectors a and b are perpendicular to both the vectors c and d.
(B) a and b are perpendicular to each other and c and d are perpendicular to each other.
(C) a and c are perpendicular to each other and b and d are perpendicular to each other.
(D) The plane spanned by a and b is perpendicular to the plane spanned by c and d.
(E) a, b, c, and d are all coplanar.

Answer: Option (E)

Explanation: Since $a \times b$ and $c \times d$ are both nonzero, but their cross product is zero, we conclude that they are both scalar multiples of each other. In particular, they are in the same line. Further, we also obtain that a, b, c, and d are individually nonzero.

We know that a and b both lie in the plane orthogonal to $a \times b$. Similarly, c and d both lie in the plane orthogonal to $c \times d$. Because $a \times b$ and $c \times d$ are in the same line, we obtain that, in fact, the plane of a and b is the same as the plane of c and d.

Performance review: 21 out of 24 people got this. 2 chose (D), 1 chose (A).

Historical note (last year): 9 out of 23 people got this correct. 6 chose (D), 4 chose (C), 3 chose (B), 1 chose (A).

(5) (*) The *correlation* between two vectors in $\mathbb{R}^n$ is defined as the quotient of the dot product of the vectors by the product of their lengths. Suppose the correlation between vectors a and b is x and the correlation between b and c is y, and suppose x, y are both positive. What is the maximum possible value of the correlation between a and c given this information? *Hint: Geometrically if θ_{ab} is the angle between a and b, θ_{bc} is the angle between b and c, and θ_{ac} is the angle between a and c, then $|\theta_{ab} - \theta_{bc}| \leq \theta_{ac} \leq \theta_{ab} + \theta_{bc}$.*
(A) xy
(B) $\max\{1, xy\}$
(C) $\min\{1, xy\}$
(D) $xy + \sqrt{(1-x^2)(1-y^2)}$
(E) $xy - \sqrt{(1-x^2)(1-y^2)}$

Answer: Option (D)

Explanation: We have that $\theta_{ab} = \arccos x$ and $\theta_{bc} = \arccos y$. Thus, $x = \cos\theta_{ab}$ and $y = \cos\theta_{bc}$. Further, from the given data, both angles are acute angles.

The maximum possible correlation between a and c occurs when the angle between these vectors is minimum, which happens when all three vectors are coplanar and θ_{ab} and θ_{bc} move in opposite directions, so $\theta_{ac} = |\theta_{ab} - \theta_{bc}|$. This gives:

$$\cos\theta_{ac} = \cos|\theta_{ab} - \theta_{bc}| = \cos\theta_{ab}\cos\theta_{bc} + \sin\theta_{ab}\sin\theta_{bc}$$

Using $\sin\theta_{ab} = \sqrt{1-x^2}$ and $\sin\theta_{bc} = \sqrt{1-y^2}$, we get the result indicated.

Further note: The *expected* correlation between a and c is xy, and this occurs roughly if the correlation between a and b is not correlated to the correlation between b and c, which basically occurs when the plane of a and b is orthogonal to the plane of b and c. The maximum correlation is in the situation described above. The minimum correlation is when a, b, and c are coplanar and θ_{ab} and θ_{bc} go in the same direction. In this case, the correlation is $\cos(\theta_{ab} + \theta_{bc}) = xy - \sqrt{(1-x^2)(1-y^2)}$. Note that the minimum correlation case changes somewhat if $\theta_{ab} + \theta_{bc} > \pi$, because in that case, the minimum correlation is -1. But that case does not occur here because both angles are acute.

Performance review: 3 out of 24 people got this. 14 chose (E), 4 chose (B), 2 chose (A), 1 chose (C).

Historical note (last year): 5 out of 23 people got this correct. 7 chose (E), 7 chose (B), 2 each chose (A) and (C).

(6) If the correlation between nonzero vector v and nonzero vector w in $\mathbb{R}^n$ is c, then we say that the *proportion* of vector w *explained by* vector v is c^2. If $v_1, v_2, \dots, v_k$ are all pairwise orthogonal nonzero vectors, and c_i is the correlation between v_i and w, then $c_1^2 + c_2^2 + \cdots + c_k^2 \leq 1$, with equality occurring if and only if $k = n$. (This is all a result of the Pythagorean theorem). If $k < n$, then $1 - (c_1^2 + c_2^2 + \cdots + c_k^2)$ is the *unexplained proportion* of w.

Suppose w is the *variation of beauty* vector, v_1 is the *variation of genes* vector, and v_2 is the *variation of make-up* vector. Assume that v_1 and v_2 are orthogonal (i.e., there is no correlation between genes and make-up choice). If the correlation between v_1 and w is 0.6 and the correlation between v_2 and w is 0.3, what proportion of the variation of beauty remains unexplained (i.e., is not explained by either genes or make-up)?

(A) 0.1
(B) 0.19
(C) 0.55
(D) 0.74
(E) 1

Answer: Option (C)

Explanation: We use the formula $1 - (0.6)^2 - (0.3)^2 = 1 - (0.36) - (0.09) = 0.55$.

In other words, genes explain 36% of the variance in beauty, make-up explains 9% of the variance, and the unexplained variance is 55%.

Note: The correlation values are bogus, this isn't a real world problem.

Performance review: All 24 people got this.

Historical note (last year): 17 out of 23 people got this correct. 4 chose (A), 1 chose (B), and 1 left the question blank.

TAKE-HOME CLASS QUIZ SOLUTIONS: DUE WEDNESDAY JANUARY 23: VECTORS, 3D, AND PARAMETRIC STUFF – MISCELLANEA

MATH 195, SECTION 59 (VIPUL NAIK)

1. Performance review

23 people took this 12-question quiz. The score distribution was as follows:

- Score of 7: 2 people.
- Score of 8: 4 people.
- Score of 9: 5 people.
- Score of 10: 9 people.
- Score of 11: 3 people.

The question-wise answers and performance review were as follows:

(1) Option (E): 20 people.
(2) Option (B): 23 people.
(3) Option (E): 22 people.
(4) Option (E): 20 people.
(5) Option (D): 21 people.
(6) Option (C): 23 people.
(7) Option (E): 12 people.
(8) Option (E): 20 people.
(9) Option (A): 10 people.
(10) Option (D): 22 people.
(11) Option (D): 7 people.
(12) Option (A): 14 people.

2. Solutions

(1) Suppose we are given three subsets Γ_1, Γ_2, and Γ_3 of $\mathbb{R}^3$ where Γ_1 is the set of solutions to $F_1(x, y, z) = 0$, Γ_2 is the set of solutions to $F_2(x, y, z) = 0$, and Γ_3 is the set of solutions to $F_3(x, y, z) = 0$. Which of the following equations gives precisely the set of points that lie in *at least two* of the subsets Γ_1, Γ_2, and Γ_3?
(A) $F_1(x, y, z)F_2(x, y, z)F_3(x, y, z) = 0$
(B) $(F_1(x, y, z))^2 + (F_2(x, y, z))^2 + (F_3(x, y, z))^2 = 0$
(C) $(F_1(x, y, z) + F_2(x, y, z) + F_3(x, y, z))^2 = 0$
(D) $(F_1(x, y, z)F_2(x, y, z)) + (F_2(x, y, z)F_3(x, y, z)) + (F_3(x, y, z)F_1(x, y, z)) = 0$
(E) $(F_1(x, y, z)F_2(x, y, z))^2 + (F_2(x, y, z)F_3(x, y, z))^2 + (F_3(x, y, z)F_1(x, y, z))^2 = 0$

Answer: Option (E)

Explanation: Option (E) means that all the statements $F_1(x, y, z)F_2(x, y, z) = 0$, $F_2(x, y, z)F_3(x, y, z) = 0$, and $F_3(x, y, z)F_1(x, y, z) = 0$ must simultaneously be true. This forces at least two of the values $F_1(x, y, z)$, $F_2(x, y, z)$, and $F_3(x, y, z)$ to equal zero. Conversely, if two or more of these are zero, so are all the pair products. Thus, a point satisfies this equation if and only if it lies in at least two of the three subsets Γ_1, Γ_2, Γ_3.

Performance review: 20 out of 23 got this. 2 chose (B), 1 chose (D).

(2) Suppose we are given three subsets Γ_1, Γ_2, and Γ_3 of $\mathbb{R}^3$ where Γ_1 is the set of solutions to $F_1(x, y, z) = 0$, Γ_2 is the set of solutions to $F_2(x, y, z) = 0$, and Γ_3 is the set of solutions to $F_3(x, y, z) = 0$. Which of the following equations gives precisely the set $\Gamma_1 \cap (\Gamma_2 \cup \Gamma_3)$?
(A) $F_1(x, y, z) + F_2(x, y, z)F_3(x, y, z) = 0$

(B) $(F_1(x,y,z))^2 + (F_2(x,y,z)F_3(x,y,z))^2 = 0$
(C) $(F_1(x,y,z) + F_2(x,y,z)F_3(x,y,z))^2 = 0$
(D) $F_1(x,y,z)(F_2(x,y,z) + F_3(x,y,z))^2 = 0$
(E) $F_1(x,y,z)((F_2(x,y,z))^2 + (F_3(x,y,z))^2) = 0$

Answer: Option (B)

Explanation: Option (B) involves a sum of two squares being zero, so both the things being squared must equal zero. Thus, $F_1(x,y,z) = 0$ and $F_2(x,y,z)F_3(x,y,z) = 0$. The solution set to the latter is the union of the solution sets for F_2 and F_3, so is $\Gamma_2 \cup \Gamma_3$. So the overall solution set is $\Gamma_1 \cap (\Gamma_2 \cup \Gamma_3)$.

Performance review: All 23 got this correct.

(3) Suppose we are given three subsets Γ_1, Γ_2, and Γ_3 of $\mathbb{R}^3$ where Γ_1 is the set of solutions to $F_1(x,y,z) = 0$, Γ_2 is the set of solutions to $F_2(x,y,z) = 0$, and Γ_3 is the set of solutions to $F_3(x,y,z) = 0$. Which of the following equations gives precisely the set $\Gamma_1 \cup (\Gamma_2 \cap \Gamma_3)$?
(A) $F_1(x,y,z) + F_2(x,y,z)F_3(x,y,z) = 0$
(B) $(F_1(x,y,z))^2 + (F_2(x,y,z)F_3(x,y,z))^2 = 0$
(C) $(F_1(x,y,z) + F_2(x,y,z)F_3(x,y,z))^2 = 0$
(D) $F_1(x,y,z)(F_2(x,y,z) + F_3(x,y,z))^2 = 0$
(E) $F_1(x,y,z)((F_2(x,y,z))^2 + (F_3(x,y,z))^2) = 0$

Answer: Option (E)

Explanation: The solution set to Option (E) is the union of the solution sets for F_1 and for $F_2^2 + F_3^2$. The latter is precisely the set of points for which $F_2 = F_3 = 0$, so is $\Gamma_2 \cap \Gamma_3$. The overall solution is thus $\Gamma_1 \cup (\Gamma_2 \cap \Gamma_3)$.

Performance review: 22 out of 23 got this. 1 chose (D).

(4) Start with two vectors a and b in $\mathbb{R}^3$ such that $a \times b \neq 0$. Consider a sequence of vectors $c_1, c_2, \dots, c_n, \dots$ in $\mathbb{R}^3$ (note: each c_n is a three-dimensional vector) defined as follows: $c_1 = a \times b$ and $c_{n+1} = a \times c_n$ for $n \geq 1$. Which *one* of the following statements is **false** about the c_ns? (5 points)
(A) All the vectors c_n are nonzero vectors.
(B) c_n and c_{n+1} are orthogonal for every n.
(C) c_n and c_{n+2} are parallel for every n.
(D) c_n and a are orthogonal for every n.
(E) c_n and b are orthogonal for every n.

Answer: Option (E)

Explanation: Note first that since $a \times b \neq 0$, both a and b are nonzero vectors.

In fact, although c_1 is orthogonal to both a and b, c_2, being orthogonal to c_1 and a, is in the plane of a and b and is orthogonal to a. Since a and b are not parallel, c_2 is not orthogonal to b.

For the other options:

Options (A) and (D): c_1 is orthogonal to a because it is a cross product involving a and a nonzero vector. At each stage, we are taking a cross product of a nonzero vector orthogonal to a with the nonzero vector a, so we get a nonzero vector orthogonal to a.

Option (B): c_{n+1} is a cross product of a and c_n, and a and c_n are both nonzero, so c_{n+1} is orthogonal to c_n.

Option (C): All the c_ns are orthogonal to a, so they are all in the plane orthogonal to a. Within this plane, each is perpendicular to its predecessor. Thus, c_n and c_{n+2} must be collinear. In fact, they points in opposite directions to each other, but are in the same line.

Performance review: 20 out of 23 got this. 2 chose (B), 1 chose (D).

Historical note (last time): 9 out of 23 people got this correct. 5 chose (C), 4 chose (D), 3 chose (A), 2 chose (B).

(5) As a general rule, what would you expect should be the dimensionality of the set of solutions to m independent and consistent equations in n variables? By solution, we mean here that the solution should be the n-tuples with coordinates in $\mathbb{R}$ (or elements of $\mathbb{R}^n$) that satisfy all the m equations. Assume $n \geq m \geq 1$.
(A) n

(B) m
(C) $n-1$
(D) $n-m$
(E) 1

Answer: Option (D)

Explanation: As a general rule, we start out with the whole space, and each new constraint, if independent of prior constraints, whittles down the dimension by 1. Thus, introducing m constraints in n-dimensional space gives a dimension of $n-m$.

Note that this is not a hard-and-fast rule, because we can use tricks like the *sum of squares* trick to combine multiple equations into a single equation. However, it is a good rule of thumb for generic equations.

Performance review: 21 out of 23 got this. 2 chose (B).

(6) As a general rule, what would you expect should be the dimensionality of the set of points in $\mathbb{R}^n$ that satisfy at least one of m independent and consistent equations in n variables? Assume $n \geq m \geq 1$.
(A) n
(B) m
(C) $n-1$
(D) $n-m$
(E) 1

Answer: Option (C)

Explanation: We get a union of solution sets for equations, and each of these solution sets is of dimension $n-1$ (since it is obtained by imposing a single constraint on n-dimensional space). The union should thus also have dimension $n-1$.

Performance review: All 23 got this correct.

(7) Measuring time t in seconds since the beginning of the year 2013, and stock prices on a 24×7 stock exchange in predetermined units, the stock prices of companies A, B, and C were found to be given by $30 + t/5000000 - \sin(t/10000)$, $16 + 7t/3000000$, and $40 + t/1000000 - 5\sin(t/10000)$. To what extent can we deduce the stock prices of the companies from each other at a given point in time, without knowing what the time is?
(A) The stock price of any of the three companies can be used to deduce the other stock prices.
(B) The stock price of company A can be used to deduce the stock prices of companies B and C, but no other deductions are possible.
(C) The stock price of company A can be used to deduce the stock prices of companies B and C, and the stock price of company C can be used to deduce the stock prices of companies A and B.
(D) The stock price of company B can be used to determine the stock prices of companies A and C, and no other deductions are possible.
(E) The stock price of company B can be used to determine the stock prices of companies A and C, and the stock prices of companies A and C can be used to deduce each other but cannot be used to uniquely deduce the stock price of company B.

Answer: Option (E)

Explanation: Since the stock price of company B is linear in t, we can determine t from it, and use this to determine the stock prices of A and C. This means that the only options we need to consider are (D) and (E).

The stock prices of A and C are related by a linear relation $5(A-30) = C-40$, or equivalently, $5A = C + 110$. Thus, either is expressible in terms of the other.

Performance review: 12 out of 23 got this correct. 10 chose (D), 1 chose (C).

(8) Lushanna is coaching 30 young athletes for a 100 meter sprint. Every day, at the beginning of the day, she asks the athlete to run 100 meters as fast as they can and notes the time taken. She thus gets a vector with 30 coordinates (measuring the time taken by all the athletes) everyday. Lushanna then plots a graph in thirty-dimensional space that includes all the points for her daily measurements. Each of the following is a sign that Lushanna's young athletes are improving. Which of these signs is **strongest**, in the sense that it would imply all the others?

(A) The norm (length) of the vector every day (after the first) is less than the norm of the vector the previous day.
(B) The sum of the coordinates of the vector every day (after the first) is less than the sum of the coordinates of the vector the previous day.
(C) The minimum of the coordinates of the vector every day (after the first) is less than the minimum of the coordinates of the vector the previous day.
(D) The maximum of the coordinates of the vector every day (after the first) is less than the maximum of the coordinates of the vector the previous day.
(E) Each of the coordinates of the vector every day (after the first) is less than the corresponding coordinate of the vector the previous day.

Answer: Option (E)

Explanation: If each of the coordinates goes down, then all the measures (the maximum, minimum and various average measures) go down. However, it is possible for any one of these measures to go down while the others don't.

Performance review: 20 out of 23 got this. 3 chose (B).

(9) In a closed system (no mass exchanged with the surroundings) a reversible chemical reaction $A+B \to C+D$, and its reverse, are proceeding. There are no other chemicals in the system, and no other reactions are proceeding in the system. A chemist studying the reaction decides to track the masses of A, B, C, and D in the system as a function of time, and plots a parametric curve in four-dimensional space. What can we say about the nature of the curve, ignoring the parametrization (i.e., just looking at the set of points covered)?
(A) It is a part of a straight line.
(B) It is a part of a circle.
(C) It is a part of a parabola.
(D) It is a part of an astroid.
(E) It is a part of a cissoid.

Answer: Option (A)

Explanation: This follows from the law of constant proportions in chemistry! Basically, the amounts of gain/loss in each coordinate are fixed in proportion based on the stoichiometry of the reaction.

Performance review: 10 out of 23 got this correct. 10 chose (B), 2 chose (D), 1 chose (C).

Lobbying special: Casa is a lobbyist for a special interest group. There are three politicians P_1, P_2, P_3 competing for a general election. Casa has computed that the probabilities of the politicians winning are p_1 for P_1, p_2 for P_2, and p_3 for P_3, with $p_1, p_2, p_3 \in [0,1]$ and $p_1 + p_2 + p_3 = 1$. Casa estimates a payoff of m_1 money units to her special interest group if P_1 wins, m_2 money units if P_2 wins, and m_3 money units if P_3 wins. (These payoffs may be in terms of passage of favorable laws, repeal of unfavorable laws, or enforcement of laws unfavorable to competitors).

(10) What is the expected payoff to the special interest group that Casa represents?
(A) $m_1 + m_2 + m_3$
(B) $(m_1 + m_2 + m_3)/3$
(C) $(p_1 + p_2 + p_3)(m_1 + m_2 + m_3)$
(D) $p_1 m_1 + p_2 m_2 + p_3 m_3$
(E) $\sqrt{m_1^2 + m_2^2 + m_3^2}$

Answer: Option (D)

Explanation: The expected payoff contributions for each victory are $p_1 m_1$, $p_2 m_2$, and $p_3 m_3$ respectively.

Performance review: 22 out of 23 got this. 1 chose (C).

(11) Casa has discovered that the bribe multipliers of the politicians are the positive reals b_1, b_2, and b_3 respectively. In other words, if Casa donates u_i money units to P_i, then the expected payoff from politician P_i winning is now $m_i + b_i u_i$. Consider the vectors $p = \langle p_1, p_2, p_3 \rangle$, $m = \langle m_1, m_2, m_3 \rangle$, $c = \langle p_1 b_1, p_2 b_2, p_3 b_3 \rangle$, and $f = \langle p_1/b_1, p_2/b_2, p_3/b_3 \rangle$ and let $u = \langle u_1, u_2, u_3 \rangle$ be the vector of the bribe quantities Casa gives to the politicians respectively. Assume that bribing politicians does not affect the relative probabilities of winning the election. Which of the following describes Casa's expected

payoff from the election, once the bribe is made (if you want to include the cost of bribes, you'd need to subtract $u_1 + u_2 + u_3$ from this answer, but we're not doing that. *Note: Some of the answer options may not make sense from a dimensions/units viewpoint, but the correct answer does make sense.*

(A) $p \cdot (m + u)$
(B) $p \cdot (m + (b \cdot u))$
(C) $p \cdot (m + (f \cdot u))$
(D) $(p \cdot m) + (c \cdot u)$
(E) $p \cdot (f \cdot m + u)$

Answer: Option (D)

Explanation: The expected payoff for each politician is $m_i + b_i u_i$, so the expected payoff accounting for probability is $p_i m_i + p_i b_i u_i = p_i m_i + c_i u_i$. The sum thus becomes $(p \cdot m) + (c \cdot u)$.

Performance review: 7 out of 23 got this. 14 chose (B), 1 chose (E), 1 left the question blank.

(12) Continuing with the full setup of the preceing question, what is Casa's optimal bribing strategy on a fixed budget of money to be used for bribes?

(A) Donate all the money to the politician with the maximum value of $p_i b_i$, i.e., to the politician corresponding to the largest coordinate of the vector c.
(B) Donate all the money to the politician with the minimum value of $p_i b_i$, i.e., to the politician corresponding to the smallest coordinate of the vector c.
(C) Donate all the money to the politician with the maximum value of p_i / b_i, i.e., to the politician corresponding to the largest coordinate of the vector f.
(D) Donate all the money to the politician with the minimum value of p_i / b_i, i.e., to the politician corresponding to the smallest coordinate of the vector f.
(E) Split the bribery budget between the politicians in the ratio $p_1 b_1 : p_2 b_2 : p_3 b_3$.

Answer: Option (A)

Explanation: $p_i b_i$ is the overall return on investment multiplier for bribing any politician. It makes sense to spend scarce bribery resources on the politician with the highest return on investment.

Performance review: 14 out of 23 got this. 6 chose (E), 3 chose (C).

TAKE-HOME CLASS QUIZ SOLUTIONS: FRIDAY JANUARY 25: LIMITS, CONTINUITY, DIFFERENTIATION REVIEW

MATH 195, SECTION 59 (VIPUL NAIK)

Note: Please see somewhat updated/improved language for Q9 – I should have stated clearly in the question that you should see options (D) and (E) before answering. However, doing so would not have improved your score, as (D) is an incorrect option.

1. Performance review

24 people took this 10 question quiz. The score distribution was as follows:

- score of 2: 2 people.
- Score of 3: 3 people.
- Score of 4: 1 person.
- Score of 5: 2 people.
- Score of 6: 1 person.
- Score of 8: 2 people.
- Score of 10: 13 people.

The question-wise answers and performance summary are below:

(1) Option (B): 17 people.
(2) Option (B): 20 people.
(3) Option (B): 23 people.
(4) Option (D): 14 people.
(5) Option (D): 18 people.
(6) Option (B): 17 people.
(7) Option (B): 21 people.
(8) Option (A): 18 people.
(9) Option (C): 16 people.
(10) Option (A): 15 people.

2. Solutions

(1) Which of the following statements is **always true**? *Earlier scores:* 2/11*,* 9/16*,* 16/28
(A) The range of a continuous nonconstant function on an open bounded interval (i.e., an interval of the form (a, b)) is an open bounded interval (i.e., an interval of the form (m, M)).
(B) The range of a continuous nonconstant function on a closed bounded interval (i.e., an interval of the form $[a, b]$) is a closed bounded interval (i.e., an interval of the form $[m, M]$).
(C) The range of a continuous nonconstant function on an open interval that may be bounded or unbounded (i.e., an interval of the form (a, b),(a, ∞), $(-\infty, a)$, or $(-\infty, \infty)$), is also an open interval that may be bounded or unbounded.
(D) The range of a continuous nonconstant function on a closed interval that may be bounded or unbounded (i.e., an interval of the form $[a, b]$, $[a, \infty)$, $(-\infty, a]$, or $(-\infty, \infty)$) is also a closed interval that may be bounded or unbounded.
(E) None of the above.

Answer: Option (B)

Explanation: This is a combination of the extreme-value theorem and the intermediate-value theorem. By the extreme-value theorem, the continuous function attains a minimum value m and a maximum value M. By the intermediate-value theorem, it attains every value between m and M. Further, it can attain no other values because m is after all the minimum and M the maximum.

The other choices:

Option (A): Think of a function that increases first and then decreases. For instance, the function $f(x) := \sqrt{1 - x^2}$ on $(-1, 1)$ has range $(0, 1]$, which is not open. Or, the function $\sin x$ on the interval $(0, 2\pi)$ has range $[-1, 1]$.

Option (C): The same counterexample as for option (A) works.

Option (D): We can get counterexamples for unbounded intervals. For instance, consider the function $f(x) := 1/x$ on $[1, \infty)$. The range of this function is $(0, 1]$, which is not closed. The idea is that we make the function approach but not reach a finite value as $x \to \infty$ (we'll talk more about this when we deal with asymptotes).

Performance review: 17 out of 24 got this. 4 chose (D), 3 chose (E).

Historical note 1: 7 out of 20 students got this correct. 6 chose (D), 5 chose (C), 2 chose (A).

Historical note 2: In the last appearance in a 153 quiz, 16 out of 28 people got this correct. 7 people chose (A), 2 people each chose (C) and (E), and 1 person chose (D).

(2) For which of the following specifications is there **no continuous function** satisfying the specifications? *Earlier scores:* 7/14*,* 21/28

(A) Domain $(0, 1)$ and range $(0, 1)$
(B) Domain $[0, 1]$ and range $(0, 1)$
(C) Domain $(0, 1)$ and range $[0, 1]$
(D) Domain $[0, 1]$ and range $[0, 1]$
(E) None of the above, i.e., we can get a continuous function for each of the specifications.

Answer: Option (B)

Explanation: By the extreme value theorem, any continuous function on a closed bounded interval must attain its maximum and minimum, and hence its image cannot be an open interval.

The other choices:

For options (A) and (D), we can pick the identity functions $f(x) := x$ on the respective domains.

For option (C), we can pick the function $f(x) := \sin^2(2\pi x)$ on the domain $(0, 1)$.

Performance review: 20 out of 24 got this. 2 chose (E), 1 each chose (A) and (C).

Historical note 1: 7 out of 20 people got this correct. 7 chose (E), 4 chose (C), 1 each chose (A) and (D).

Historical note 2: In the last appearance in a 153 quiz, 21 out of 28 people got this correct. 5 people chose (C) and 2 people chose (E).

(3) Suppose f is a continuously differentiable function from the open interval $(0, 1)$ to $\mathbb{R}$. Suppose, further, that there are exactly 14 values of c in $(0, 1)$ for which $f(c) = 0$. What can we say is **definitely true** about the number of values of c in the open interval $(0, 1)$ for which $f'(c) = 0$? *Earlier scores:* 7/15*,* 19/28

(A) It is at least 13 and at most 15.
(B) It is at least 13, but we cannot put any upper bound on it based on the given information.
(C) It is at most 15, but we cannot put any lower bound (other than the meaningless bound of 0) based on the given information.
(D) It is at most 13.
(E) It is at least 15.

Answer: Option (B)

Explanation: Suppose the zeros of f are $a_1 < a_2 < \cdots < a_{14}$. By Rolle's theorem, there is *at least one* zero of f' between each a_i and a_{i+1}. There may be more, since Rolle's theorem gives only a *lower* bound. This gives thirteen solutions c to $f'(c) = 0$.

Note that in order to apply Rolle's theorem, it is enough to be given that f is differentiable, so the additional hypothesis that f' is continuous is not necessary.

Performance review: 23 out of 24 got this. 1 chose (A).

Historical note 1: 12 out of 20 people got this correct. 4 chose (A), 2 chose (E), 1 each chose (C) and (D).

Historical note 2: In the last appearance in a 153 quiz, 19 out of 28 people got this correct. 5 people chose (A), 2 people chose (C), and 1 person each chose (D) and (E).

(4) Consider the function $f(x) := \{ \begin{array}{cc} x, & 0 \le x \le 1/2 \\ x - (1/7), & 1/2 < x \le 1 \end{array}$. Define by $f^{[n]}$ the function obtained by iterating f n times, i.e., the function $f \circ f \circ f \circ \cdots \circ f$ where f occurs n times. What is the smallest n for which $f^{[n]} = f^{[n+1]}$? *Earlier score:* 3/16
(A) 1
(B) 2
(C) 3
(D) 4
(E) 5
Answer: Option (D)
Explanation: We need to iterate f enough times that everything gets inside $[0, 1/2]$, after which it becomes stable. Note that each time, the value goes down by 1/7. Thus, for any $x \le 1$, we need at most four steps to bring it in $[0, 1/2]$, with the upper bound of 4 being attained for 1.
Performance review: 14 out of 24 got this. 5 chose (C), 4 chose (A), 1 chose (E).
Historical note 1: 4 out of 20 people got this correct. 7 chose (B), 7 chose (C), 1 each chose (A) and (E).
Historical note 2: In the last appearance in a 153 quiz, 10 out of 28 people got this correct. 9 people chose (C), 5 people chose (B), 2 people chose (A), and 2 people left the question blank.

(5) Suppose f and g are functions $(0, 1)$ to $(0, 1)$ that are both right continuous on $(0, 1)$. Which of the following is *not* guaranteed to be right continuous on $(0, 1)$? *Earlier scores:* 3/11*,* 9/14*,* 20/28
(A) $f + g$, i.e., the function $x \mapsto f(x) + g(x)$
(B) $f - g$, i.e., the function $x \mapsto f(x) - g(x)$
(C) $f \cdot g$, i.e., the function $x \mapsto f(x)g(x)$
(D) $f \circ g$, i.e., the function $x \mapsto f(g(x))$
(E) None of the above, i.e., they are all guaranteed to be right continuous functions
Answer: Option (D)
Explanation: See the explanation for Question 2 on the October 1 quiz. Note that that quiz uses left continuity, but the example can be adapted to right continuity.
Performance review: 18 out of 24 got this. 3 chose (B), 2 chose (E), 1 chose (C).
Historical note 1: 7 out of 20 people got this correct. 6 chose (E), 5 chose (C), 2 chose (B).
Historical note 2: In the last appearance in a 153 quiz, 20 out of 28 people got this correct. 4 people chose (C), 2 people chose (B), 1 person chose (E), and 1 person left the question blank.

(6) Suppose f and g are increasing functions from $\mathbb{R}$ to $\mathbb{R}$. Which of the following functions is *not* guaranteed to be an increasing functions from $\mathbb{R}$ to $\mathbb{R}$? *Earlier scores:* 1/15*,* 9/16*,* 18/28
(A) $f + g$
(B) $f \cdot g$
(C) $f \circ g$
(D) All of the above, i.e., none of them is guaranteed to be increasing.
(E) None of the above, i.e., they are all guaranteed to be increasing.
Answer: Option (B)
Explanation: The problem with option (B) arises when one or both functions take negative values. For instance, consider the case $f(x) := x$ and $g(x) := x$. Both are increasing functions on all of $\mathbb{R}$. However, the pointwise product is the function $x \mapsto x^2$, which is a decreasing function for negative x.

Formally, the issue is that we cannot multiply inequalities of the form $A < B$ and $C < D$ unless we are guaranteed to be working with positive numbers.

The other choices:

Option (A): For any $x_1 < x_2$, we have $f(x_1) < f(x_2)$ and $g(x_1) < g(x_2)$. Adding up, we get $f(x_1) + g(x_1) < f(x_2) + g(x_2)$, so $(f + g)(x_1) < (f + g)(x_2)$.

Option (C): For any $x_1 < x_2$, we have $g(x_1) < g(x_2)$ since g is increasing. Now, we use the factthat f is increasing to compare its values at the two points $g(x_1)$ and $g(x_2)$, and we get $f(g(x_1)) < f(g(x_2))$. We thus get $(f \circ g)(x_1) < (f \circ g)(x_2)$.

Performance review: 17 out of 24 got this. 4 chose (E), 1 each chose (A), (C), and (D).

Historical note 1: 8 out of 20 people got this correct. 7 chose (E), 4 chose (C), 1 chose (D).

Historical note 2: In the last appearance in a 153 quiz, 18 out of 28 people got this correct. 6 people chose (E) and 4 people chose (C).

(7) Suppose F and G are two functions defined on $\mathbb{R}$ and k is a natural number such that the k^{th} derivatives of F and G exist and are equal on all of $\mathbb{R}$. Then, $F-G$ must be a polynomial function. What is the **maximum possible degree** of $F-G$? (Note: Assume constant polynomials to have degree zero) *Earlier score:* 6/16, 10/28

(A) $k-2$
(B) $k-1$
(C) k
(D) $k+1$
(E) There is no bound in terms of k.

Answer: Option (B)

Explanation: F and G having the same k^{th} derivative is equivalent to requiring that $F-G$ have k^{th} derivative equal to zero. For $k=1$, this gives constant functions (polynomials of degree 0). Each time we increment k, the degree of the polynomial could potentially go up by 1. Thus, the answer is $k-1$.

Performance review: 21 out of 24 got this. 2 chose (A), 1 chose (E).

Historical note 1: 8 out of 20 people got this correct. 6 chose (D), 3 chose (C), 2 chose (E), 1 chose (A).

Historical note 2: In the last appearance in a 153 quiz, 10 out of 28 people got this correct. 5 people each chose (D) and (E) and 4 people each chose (A) and (C).

Action point: This is a question you really *should* get correct!

(8) Suppose f is a continuous function on $\mathbb{R}$. Clearly, f has antiderivatives on $\mathbb{R}$. For all but one of the following conditions, it is possible to guarantee, without any further information about f, that there exists an antiderivative F satisfying that condition. **Identify the exceptional condition** (i.e., the condition that it may not always be possible to satisfy). *Earlier score:* 3/16, 10/28

(A) $F(1) = F(0)$.
(B) $F(1) + F(0) = 0$.
(C) $F(1) + F(0) = 1$.
(D) $F(1) = 2F(0)$.
(E) $F(1)F(0) = 0$.

Answer: Option (A)

Explanation: Suppose G is an antiderivative for f. The general expression for an antiderivative is $G + C$, where C is constant. We see that for options (b), (c), and (d), it is always possible to solve the equation we obtain to get one or more real values of C. However, (a) simplifies to $G(1) + C = G(0) + C$, whereby C is canceled, and we are left with the statement $G(1) = G(0)$. If this statement is true, then *all* choices of C work, and if it is false, then *none* works. Since we cannot guarantee the truth of the statement, (a) is the exceptional condition.

Another way of thinking about this is that $F(1) - F(0) = \int_0^1 f(x)\,dx$, regardless of the choice of F. If this integral is 0, then any antiderivative works. If it is not zero, no antiderivative works.

Performance review: 18 out of 24 got this. 3 chose (C), 2 chose (E), 1 chose (B).

Historical note 1: 2 out of 20 people got this correct. 10 chose (E), 4 chose (D), 3 chose (C), 1 chose (B).

Historical note 2: In the last appearance in a 153 quiz, 10 out of 28 people got this correct. 6 people chose (B), 5 people chose (E), 4 people chose (D), 2 people chose (C), and 1 person left the question blank.

(9) Suppose F is a function defined on $\mathbb{R} \setminus \{0\}$ such that $F'(x) = -1/x^2$ for all $x \in \mathbb{R} \setminus \{0\}$. Which of the following pieces of information is/are **sufficient** to determine F completely? Please see Options (D) and (E) before answering.

(A) The value of F at any two positive numbers.
(B) The value of F at any two negative numbers.
(C) The value of F at a positive number and a negative number.

(D) Any of the above pieces of information is sufficient, i.e., we need to know the value of F at any two numbers.

(E) None of the above pieces of information is sufficient.

Answer: Option (C)

Explanation: There are two open intervals: $(-\infty, 0)$ and $(0, \infty)$, on which we can look at F. On each of these intervals, $F(x) = 1/x+$ a constant, but the constant for $(-\infty, 0)$ may differ from the constant for $(0, \infty)$. Thus, we need the initial value information at one positive number and one negative number.

Performance review: 16 out of 24 got this. 8 chose (D).

Historical note 1: 5 out of 20 people got this correct. 12 chose (D), 3 chose (A).

Historical note 2: In the last appearance in a 153 quiz, 15 out of 28 people got this correct. 9 people chose (D), 2 people chose (E), and 1 person each chose (A) and (B).

(10) Suppose F and G are continuously differentiable functions on all of $\mathbb{R}$ (i.e., both F' and G' are continuous). Which of the following is **not necessarily true**? *Earlier scores:* 0, 10/16

(A) If $F'(x) = G'(x)$ for all integers x, then $F - G$ is a constant function when restricted to integers, i.e., it takes the same value at all integers.

(B) If $F'(x) = G'(x)$ for all numbers x that are not integers, then $F - G$ is a constant function when restricted to the set of numbers x that are not integers.

(C) If $F'(x) = G'(x)$ for all rational numbers x, then $F - G$ is a constant function when restricted to the set of rational numbers.

(D) If $F'(x) = G'(x)$ for all irrational numbers x, then $F - G$ is a constant function when restricted to the set of irrational numbers.

(E) None of the above, i.e., they are all necessarily true.

Answer: Option (A).

Explanation: The fact that the derivatives of two functions agree at integers says nothing about how the derivatives behave elsewhere – they could differ quite a bit at other places. Hence, (A) is not necessarily true, and hence must be the right option. All the other options are correct as statements and hence cannot be the right option. This is because in all of them, the set of points where the derivatives agree is *dense* – it intersects every open interval. So, continuity forces the functions F' and G' to be equal everywhere, forcing $F - G$ to be constant everywhere.

Performance review: 15 out of 24 got this. 6 chose (E), 3 chose (B).

Historical note 1: 7 out of 20 people got this correct. 7 chose (E), 3 chose (D), 3 chose (C).

Historical note 2: In the last appearance in a 153 quiz, 11 out of 28 people got this correct. 8 people chose (E), 5 people chose (C), and 2 people each chose (B) and (D).

TAKE-HOME CLASS QUIZ SOLUTIONS: DUE FRIDAY FEBRUARY 1: LIMITS

MATH 195, SECTION 59 (VIPUL NAIK)

1. Performance review

27 people took this 14-question quiz. The score distribution was as follows:

- Score of 8: 2 people.
- Score of 9: 3 people.
- Score of 10: 2 people.
- Score of 11: 4 people.
- Score of 12: 5 people.
- Score of 13: 5 people.
- Score of 14: 6 people.

The question wise answers and performance review were as follows:

(1) Option (A): All 27 people.
(2) Option (D): 22 people.
(3) Option (C): 26 people.
(4) Option (C): 20 people.
(5) Option (B): 21 people.
(6) Option (C): 23 people.
(7) Option (B): 26 people.
(8) Option (C): 26 people.
(9) Option (E): 21 people.
(10) Option (D): 21 people.
(11) Option (A): 18 people.
(12) Option (E): 22 people.
(13) Option (A): 19 people.
(14) Option (B): 24 people.

2. Solutions

(1) We call a function f left continuous on an open interval I if, for all $a \in I$, $\lim_{x \to a^-} f(x) = f(a)$. Which of the following is an example of a function that is left continuous but not continuous on $(0, 1)$? If all are examples, please select Option (E).

(A) $f(x) := \begin{cases} x, & 0 < x \le 1/2 \\ 2x, & 1/2 < x < 1 \end{cases}$

(B) $f(x) := \begin{cases} x, & 0 < x < 1/2 \\ 2x, & 1/2 \le x < 1 \end{cases}$

(C) $f(x) := \begin{cases} x, & 0 < x \le 1/2 \\ 2x - (1/2), & 1/2 < x < 1 \end{cases}$

(D) $f(x) := \begin{cases} x, & 0 < x < 1/2 \\ 2x - (1/2), & 1/2 \le x < 1 \end{cases}$

(E) All of the above

Answer: Option (A)

Explanation: Note that in all four cases, the two pieces of the function are continuous. Thus, the relevant questions are: (i) do the two definitions agree at the point where the definition changes (in all four cases here, $1/2$)? and (ii) is the point (in all cases, $1/2$) where the definition changes included in the left or the right piece?

For options (C) and (D), the definitions on the left and right piece agree at $1/2$. Namely the function x and $2x - (1/2)$ both take the value $1/2$ at the domain point $1/2$. Thus, options (C) and (D) both define continuous functions (in fact, the same continuous function).

This leaves options (A) and (B). For these, the left definition x and the right definition $2x$ do not match at $1/2$: the former gives $1/2$ and the latter gives 1. In other words, the function has a jump discontinuity at $1/2$. Thus, (ii) becomes relevant: is $1/2$ included in the left or the right definition?

For option (A), $1/2$ is included in the left definition, so $f(1/2) = 1/2 = \lim_{x \to 1/2^-} f(x)$. On the other hand, $\lim_{x \to 1/2^+} f(x) = 1$. Thus, the f in option (A) is left continuous but not right continuous.

For option (B), $1/2$ is included in the right definition, so $f(1/2) = 1$ and f is right continuous but not left continuous at $1/2$.

Performance review: All 27 got this.

Historical note (last time): 47 out of 49 people got this correct. 1 person each chose (B) and (C).

(2) Suppose f and g are functions $(0, 1)$ to $(0, 1)$ that are both left continuous on $(0, 1)$. Which of the following is *not* guaranteed to be left continuous on $(0, 1)$? Please see Option (E) before answering.

(A) $f + g$, i.e., the function $x \mapsto f(x) + g(x)$
(B) $f - g$, i.e., the function $x \mapsto f(x) - g(x)$
(C) $f \cdot g$, i.e., the function $x \mapsto f(x)g(x)$
(D) $f \circ g$, i.e., the function $x \mapsto f(g(x))$
(E) None of the above, i.e., they are all guaranteed to be left continuous functions

Answer: Option (D)

Explanation: We need to construct an explicit example, but we first need to do some theoretical thinking to motivate the right example. The full reasoning is given below.

Motivation for example: Left hand limits split under addition, subtraction and multiplication, so options (A)-(C) are guaranteed to be left continuous, and are thus false. This leaves the option $f \circ g$ for consideration. Let us look at this in more detail.

For $c \in (0, 1)$, we want to know whether:

$$\lim_{x \to c^-} f(g(x)) \stackrel{?}{=} f(g(c))$$

We do know, by assumption, that, as x approaches c from the left, $g(x)$ approaches $g(c)$. However, we do not know whether $g(x)$ approaches $g(c)$ from the left or the right or in oscillatory fashion. If we could somehow guarantee that $g(x)$ approaches $g(c)$ from the left, then we would obtain that the above limit holds. However, the given data does not guarantee this, so (D) is false.

We need to construct an example where g is *not* an increasing function. In fact, we will try to pick g as a decreasing function, so that when x approaches c from the left, $g(x)$ approaches $g(c)$ from the right. As a result, when we compose with f, the roles of left and right get switched. Further, we need to construct f so that it is left continuous but not right continuous.

Explanation with example: Consider the case where, say:

$$f(x) := \begin{cases} 1/3, & 0 < x \le 1/2 \\ 2/3, & 1/2 < x < 1 \end{cases}$$

and

$$g(x) := 1 - x$$

Note that both functions have range a subset of $(0, 1)$.

Composing, we obtain that:

$$f(g(x)) = \begin{cases} 2/3, & 0 < x < 1/2 \\ 1/3, & 1/2 \le x < 1 \end{cases}$$

f is left continuous but not right continuous at $1/2$, whereas $f \circ g$ is right continuous but not left continuous at $1/2$.

Performance review: 22 out of 27 got this. 2 chose (E), 1 each chose (C), 1 chose (B).

Historical note (last time): 20 out of 49 people got this correct. 26 people chose (E), 2 chose (C), 1 chose (B).

(3) Which of these is the correct interpretation of $\lim_{x \to c} f(x) = L$ in terms of the definition of limit? Please see Option (E) before answering.
- (A) For every $\alpha > 0$, there exists $\beta > 0$ such that if $0 < |x - c| < \alpha$, then $|f(x) - L| < \beta$.
- (B) There exists $\alpha > 0$ such that for every $\beta > 0$, and $0 < |x - c| < \alpha$, we have $|f(x) - L| < \beta$.
- (C) For every $\alpha > 0$, there exists $\beta > 0$ such that if $0 < |x - c| < \beta$, then $|f(x) - L| < \alpha$.
- (D) There exists $\alpha > 0$ such that for every $\beta > 0$ and $0 < |x - c| < \beta$, we have $|f(x) - L| < \alpha$.
- (E) None of the above

Answer: Option (C)

Explanation: α plays the role of ε and β plays the role of δ.

Performance review: 26 out of 27 got this. 1 chose (B).

Historical note (last time): 44 out of 49 people got this correct. 3 people chose (A), 2 chose (B).

(4) Suppose $f : \mathbb{R} \to \mathbb{R}$ is a function. Which of the following says that f does not have a limit at any point in $\mathbb{R}$ (i.e., there is no point $c \in \mathbb{R}$ for which $\lim_{x \to c} f(x)$ exists)? If all, please select Option (E).
- (A) For every $c \in \mathbb{R}$, there exists $L \in \mathbb{R}$ such that for every $\varepsilon > 0$, there exists $\delta > 0$ such that for all x satisfying $0 < |x - c| < \delta$, we have $|f(x) - L| \geq \varepsilon$.
- (B) There exists $c \in \mathbb{R}$ such that for every $L \in \mathbb{R}$, there exists $\varepsilon > 0$ such that for every $\delta > 0$, there exists x satisfying $0 < |x - c| < \delta$ and $|f(x) - L| \geq \varepsilon$.
- (C) For every $c \in \mathbb{R}$ and every $L \in \mathbb{R}$, there exists $\varepsilon > 0$ such that for every $\delta > 0$, there exists x satisfying $0 < |x - c| < \delta$ and $|f(x) - L| \geq \varepsilon$.
- (D) There exists $c \in \mathbb{R}$ and $L \in \mathbb{R}$ such that for every $\varepsilon > 0$, there exists $\delta > 0$ such that for all x satisfying $0 < |x - c| < \delta$, we have $|f(x) - L| \geq \varepsilon$.
- (E) All of the above.

Answer: Option (C)

Explanation: Our statement should be that *every* c has no limit. In other words, for *every* c and *every* L, it is *not* true that $\lim_{x \to c} f(x) = L$. That's exactly what option (C) says.

Performance review: 20 out of 27 got this. 4 chose (A), 1 each chose (B), (D), and (E).

Historical note (last time): 20 out of 49 people got this correct. 15 chose (B), 7 chose (E), 4 chose (D), 3 chose (A).

(5) In the usual $\varepsilon - \delta$ definition of limit for a given limit $\lim_{x \to c} f(x) = L$, if a given value $\delta > 0$ works for a given value $\varepsilon > 0$, then which of the following is true? Please see Option (E) before answering.
- (A) Every smaller positive value of δ works for the same ε. Also, the given value of δ works for every smaller positive value of ε.
- (B) Every smaller positive value of δ works for the same ε. Also, the given value of δ works for every larger value of ε.
- (C) Every larger value of δ works for the same ε. Also, the given value of δ works for every smaller positive value of ε.
- (D) Every larger value of δ works for the same ε. Also, the given value of δ works for every larger value of ε.
- (E) None of the above statements need always be true.

Answer: Option (B)

Explanation: This can be understood in multiple ways. One is in terms of the prover-skeptic game. A particular choice of δ that works for a specific ε also works for larger εs, because the function is already "trapped" in a smaller region. Further, smaller choices of δ also work because the skeptic has fewer values of x.

Rigorous proofs are being skipped here, but you can review the formal definition of limit notes if this stuff confuses you.

Performance review: 21 out of 27 got this. 3 chose (E), 2 chose (D), 1 chose (C).

Historical note (last time): 31 out of 49 people got this correct. 6 each chose (A) and (E), 5 chose (C), 1 chose (D).

(6) Which of the following is a correct formulation of the statement $\lim_{x \to c} f(x) = L$, in a manner that avoids the use of εs and δs? Please see Option (E) before answering.

(A) For every open interval centered at c, there is an open interval centered at L such that the image under f of the open interval centered at c (excluding the point c itself) is contained in the open interval centered at L.

(B) For every open interval centered at c, there is an open interval centered at L such that the image under f of the open interval centered at c (excluding the point c itself) contains the open interval centered at L.

(C) For every open interval centered at L, there is an open interval centered at c such that the image under f of the open interval centered at c (excluding the point c itself) is contained in the open interval centered at L.

(D) For every open interval centered at L, there is an open interval centered at c such that the image under f of the open interval centered at c (excluding the point c itself) contains the open interval centered at L.

(E) None of the above.

Answer: Option (C)

Explanation: The "open interval centered at L" describes the "$\varepsilon > 0$" part of the definition (where the open interval is the interval $(L - \varepsilon, L + \varepsilon)$). The "open interval centered at c" describes the "$\delta > 0$" part of the definition (where the open interval is the interval $(c - \delta, c + \delta)$). x being in the open interval centered at c (except the case $x = c$) is equivalent to $0 < |x - c| < \delta$, and $f(x)$ being in the open interval centered at L is equivalent to $|f(x) - L| < \varepsilon$.

Performance review: 23 out of 27 got this. 3 chose (D), 1 chose (B).

Historical note (last time): 24 out of 49 people got this correct. 12 chose (A), 8 chose (D), 3 chose (E), 2 chose (B).

(7) Consider the function:

$$f(x) := \begin{cases} x, & x \text{ rational} \\ 1/x, & x \text{ irrational} \end{cases}$$

What is the set of all points at which f is continuous?

(A) $\{0, 1\}$

(B) $\{-1, 1\}$

(C) $\{-1, 0\}$

(D) $\{-1, 0, 1\}$

(E) f is continuous everywhere

Answer: Option (B)

Explanation: In this interesting example, instead of a *left* versus *right* split, we are splitting the domain into rationals and irrationals. For the overall limit to exist at c, we need that: (i) the limit for the function as defined for rationals exists at c, (ii) the limit for the function as defined for irrationals exists at c, and (iii) the two limits are equal.

Note that regardless of whether the point c is rational or irrational, we need *both* the rational domain limit and the irrational domain limit to exist and be equal at c. This is because rational numbers are surrounded by irrational numbers and vice versa – both rational numbers and irrational numbers are dense in the reals – hence at any point, we care about the limits restricted to the rationals as well as the irrationals.

The limit for rationals exists for all c and equals the value c. The limit for irrationals exists for all $c \neq 0$ and equals the value $1/c$. For these two numbers to be equal, we need $c = 1/c$. Solving, we get $c^2 = 1$ so $c = \pm 1$.

Performance review: 26 out of 27 got this. 1 chose (A).

Historical note (last time): 39 out of 49 people got this correct. 7 chose (D), 2 chose (A), 1 chose (E).

(8) The graph $y = f(x)$ of a function f defined on all reals has a horizontal asymptote $y = c$ as x approaches $+\infty$. Which of the following is the correct definition of this?

(A) For every $a \in \mathbb{R}$, there exists $\delta > 0$ such that for all x satisfying $0 < |x - c| < \delta$, we have $f(x) > a$.

(B) For every $a \in \mathbb{R}$, there exists $\varepsilon > 0$ such that for all x satisfying $x > a$, we have $|f(x) - c| < \varepsilon$.

(C) For every $\varepsilon > 0$, there exists $a \in \mathbb{R}$ such that for all x satisfying $x > a$, we have $|f(x) - c| < \varepsilon$.

(D) For every $\delta > 0$, there exists $a \in \mathbb{R}$ such that for all x satisfying $0 < |x - c| < \delta$, we have $f(x) > a$.

(E) For every $\varepsilon > 0$, there exists $\delta > 0$ such that for all x satisfying $0 < |x - c| < \delta$, we have $|f(x) - c| < \varepsilon$.

Answer: Option (C)

Explanation: The neighborhood of c (picked by the skeptic) is the interval $(c - \varepsilon, c + \varepsilon)$, and it is parametrized by its radius ε. The neighborhood of $+\infty$ (picked by the prover) is the interval (a, ∞), and it is parametrzed by its lower endpoint a. The skeptic then picks x in the neighborhood specified by the prover, i.e., $f(x) > a$, and then they check whether $f(x)$ is in the chosen neighborhood of c.

Performance review: 26 out of 27 got this. 1 chose (B).

Historical note (last time): 35 out of 46 got this correct. 4 each chose (B) and (E). 3 chose (A).

(9) Which of the following is the correct definition of $\lim_{x \to c^-} f(x) = -\infty$ (in words: the left hand limit of f at c is $-\infty$)?

(A) For every $a \in \mathbb{R}$, there exists $\delta > 0$ such that for all x satisfying $0 < |x - c| < \delta$, we have $f(x) > a$.

(B) For every $a \in \mathbb{R}$, there exists $\delta > 0$ such that for all x satisfying $0 < x - c < \delta$, we have $f(x) > a$.

(C) For every $a \in \mathbb{R}$, there exists $\delta > 0$ such that for all x satisfying $0 < x - c < \delta$, we have $f(x) < a$.

(D) For every $a \in \mathbb{R}$, there exists $\delta > 0$ such that for all x satisfying $0 < c - x < \delta$, we have $f(x) > a$.

(E) For every $a \in \mathbb{R}$, there exists $\delta > 0$ such that for all x satisfying $0 < c - x < \delta$, we have $f(x) < a$.

Answer: Option (E)

Explanation: The neighborhood of $-\infty$ chosen by the skeptic is $(-\infty, a)$, and it is parameterized by its upper endpoint a. The prover picks the parameter δ for the left side δ "half-neighborhood" of c, namely $(c - \delta, c)$. The skeptic then picks x in this half-neighborhood, and they then check whether $f(x) \in (-\infty, a)$. Translating the interval conditions into inequality notation, we get the definition as stated.

Performance review: 21 out of 27 got this. 3 chose (C), 2 chose (D), 1 chose (B).

Historical note (last time): 37 out of 46 got this correct. 3 each chose (B), (C), and (D).

(10) Suppose f is a function defined on all of $\mathbb{R}$ and $c \in \mathbb{R}$. Which of the following is the correct $\varepsilon - \delta$ definition for the statement "f is differentiable at c"?

(A) For every $L \in \mathbb{R}$, there exists $\varepsilon > 0$ such that for every $\delta > 0$, there exists x such that $0 < |x - c| < \delta$ and $|f(x) - f(c) - L(x - c)| \geq |x - c|\varepsilon$.

(B) For every $L \in \mathbb{R}$, there exists $\varepsilon > 0$ such that for every $\delta > 0$, there exists x such that $0 < |x - c| < \delta$ and $|f(x) - f(c) - L(x - c)| < |x - c|\varepsilon$.

(C) There exists $L \in \mathbb{R}$ such that for every $\varepsilon > 0$, there exists $\delta > 0$ such that for every x satisfying $0 < |x - c| < \delta$, we have $|f(x) - f(c) - L(x - c)| \geq |x - c|\varepsilon$

(D) There exists $L \in \mathbb{R}$ such that for every $\varepsilon > 0$, there exists $\delta > 0$ such that for every x satisfying $0 < |x - c| < \delta$, we have $|f(x) - f(c) - L(x - c)| < |x - c|\varepsilon$

(E) There exists $L \in \mathbb{R}$ such that there exists $\varepsilon > 0$ such that for every $\delta > 0$, there exists x such that $0 < |x - c| < \delta$ and $|f(x) - f(c) - L(x - c)| < |x - c|\varepsilon$.

Answer: Option (D)

Explanation: We would like to say that there exists $L \in \mathbb{R}$ (where L will be the claimed value of $f'(c)$) such that:

$$\lim_{x \to c} \frac{f(x) - f(c)}{x - c} = L$$

To do this, we must say that for every $\varepsilon > 0$, there exists $\delta > 0$ such that for all x satisfying $0 < |x - c| < \delta$, we have:

$$\left|\frac{f(x)-f(c)}{x-c}-L\right|<\varepsilon$$

Rewriting the final inequality, we get option (D).

Performance review: 21 out of 27 got this. 3 chose (B), 1 each chose (A) and (C), 1 left the question blank.

Historical note (last time): 17 out of 44 got this. 13 chose (A), 12 chose (B), 1 each chose (C) and (E).

(11) Suppose f is a function defined on all of $\mathbb{R}$ and $c \in \mathbb{R}$. Which of the following is the correct $\varepsilon-\delta$ definition for the statement "f is not differentiable at c"?
(A) For every $L \in \mathbb{R}$, there exists $\varepsilon > 0$ such that for every $\delta > 0$, there exists x such that $0 < |x-c| < \delta$ and $|f(x) - f(c) - L(x-c)| \geq |x-c|\varepsilon$.
(B) For every $L \in \mathbb{R}$, there exists $\varepsilon > 0$ such that for every $\delta > 0$, there exists x such that $0 < |x-c| < \delta$ and $|f(x) - f(c) - L(x-c)| < |x-c|\varepsilon$.
(C) There exists $L \in \mathbb{R}$ such that for every $\varepsilon > 0$, there exists $\delta > 0$ such that for every x satisfying $0 < |x-c| < \delta$, we have $|f(x) - f(c) - L(x-c)| \geq |x-c|\varepsilon$
(D) There exists $L \in \mathbb{R}$ such that for every $\varepsilon > 0$, there exists $\delta > 0$ such that for every x satisfying $0 < |x-c| < \delta$, we have $|f(x) - f(c) - L(x-c)| < |x-c|\varepsilon$
(E) There exists $L \in \mathbb{R}$ such that there exists $\varepsilon > 0$ such that for every $\delta > 0$, there exists x such that $0 < |x-c| < \delta$ and $|f(x) - f(c) - L(x-c)| < |x-c|\varepsilon$.

Answer: Option (A)

Explanation: We would like to say that for every $L \in \mathbb{R}$, the statement:

$$\lim_{x \to c} \frac{f(x)-f(c)}{x-c} = L$$

is false. To do this, we must say that there exists $\varepsilon > 0$ such that the difference quotient (i.e., the expression on the left) cannot be trapped within the interval $(L-\varepsilon, L+\varepsilon)$. In other words, for every $\delta > 0$, there is some value of x such that $0 < |x-c| < \delta$ and:

$$\left|\frac{f(x)-f(c)}{x-c}-L\right| \geq \varepsilon$$

Rewriting the final inequality, we get option (A).

Performance review: 18 out of 27 got this. 7 chose (C), 2 chose (D).

Historical note (administered in an earlier year): 5 out of 15 people got this correct. 4 people chose (C), 2 people chose (D), 2 people chose (E), 1 person chose (B), and 1 person left the question blank.

(12) Suppose $f : \mathbb{R} \to \mathbb{R}$ is a function. Identify which of these definitions is *not* correct for $\lim_{x \to c} f(x) = L$, where c and L are both finite real numbers. If all are correct, please select Option (E).
(A) For every $\varepsilon > 0$, there exists $\delta > 0$ such that if $x \in (c-\delta, c+\delta) \setminus \{c\}$, then $f(x) \in (L-\varepsilon, L+\varepsilon)$.
(B) For every $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$, there exist $\delta_1 > 0$ and $\delta_2 > 0$ such that if $x \in (c-\delta_1, c+\delta_2) \setminus \{c\}$, then $f(x) \in (L-\varepsilon_1, L+\varepsilon_2)$.
(C) For every $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$, there exists $\delta > 0$ such that if $x \in (c-\delta, c+\delta) \setminus \{c\}$, then $f(x) \in (L-\varepsilon_1, L+\varepsilon_2)$.
(D) For every $\varepsilon > 0$, there exist $\delta_1 > 0$ and $\delta_2 > 0$ such that if $x \in (c-\delta_1, c+\delta_2) \setminus \{c\}$, then $f(x) \in (L-\varepsilon, L+\varepsilon)$.
(E) None of these, i.e., all definitions are correct.

Answer: Option (E)

Explanation: Although the usual $\varepsilon-\delta$ definition uses centered intervals, i.e., intervals centered at the points c and L, this is not a necessary aspect of the definition. So, instead of taking centered intervals $(c-\delta, c+\delta)$ or $(L-\varepsilon, L+\varepsilon)$, we could consider open intervals that have different amounts on the left and on the right. Thus, all four definitions are correct.

Performance review: 22 out of 27 got this. 3 chose (D), 2 chose (C).

Historical note (last time): 33 out of 41 got this. 3 chose (C). 2 each chose (B) and (D), 1 left the question blank.

(13) In the usual $\varepsilon - \delta$ definition of limit, we find that the value $\delta = 0.2$ for $\varepsilon = 0.7$ for a function f at 0, and the value $\delta = 0.5$ works for $\varepsilon = 1.6$ for a function g at 0. What value of δ *definitely* works for $\varepsilon = 2.3$ for the function $f + g$ at 0?
(A) 0.2
(B) 0.3
(C) 0.5
(D) 0.7
(E) 0.9

Answer: Option (A)

Explanation: We choose the *smaller* of the δs to guarantee that *both* f and g are within their respective ε-distances of the targets – 0.7 in the case of f and 1.6 in the case of g. Now, the triangle inequality guarantees that $f + g$ is within 2.3 of its proposed limit.

Performance review: 19 out of 27 got this. 6 chose (D), 1 each chose (C) and (E).

Historical note (last time): 36 out of 41 got this. 3 chose (D), 1 each chose (C) and (E).

(14) The sum of limits theorem states that $\lim_{x \to c} [f(x) + g(x)] = \lim_{x \to c} f(x) + \lim_{x \to c} g(x)$ *if* the right side is defined. One of the choices below gives an example where the left side of the equality is defined and finite but the right side makes no sense. Identify the correct choice.
(A) $f(x) := 1/x$, $g(x) := -1/(x + 1)$, $c = 0$.
(B) $f(x) := 1/x$, $g(x) := (x - 1)/x$, $c = 0$.
(C) $f(x) := \arcsin x$, $g(x) := \arccos x$, $c = 1/2$.
(D) $f(x) := 1/x$, $g(x) = x$, $c = 0$.
(E) $f(x) := \tan x$, $g(x) := \cot x$, $c = 0$.

Answer: Option (B)

Explanation: $f + g$ is the constant function 1, so it has a limit. On the other hand, both f and g have one-sided limits of $\pm\infty$.

For options (A), (D), and (E), one of the function f and g has a finite limit, and the other has an infinite or undefined limit, and the sum has an infinite or undefined limit. Option (C) is a case where f, g, and $f + g$ all have finite limits.

Performance review: 24 out of 27 got this. 2 chose (A), 1 chose (E).

Historical note (last time): 36 out of 41 got this. 2 each chose (A) and (E), 1 chose (C).

CLASS QUIZ SOLUTIONS: FRIDAY FEBRUARY 1: MULTIVARIABLE FUNCTION BASICS

MATH 195, SECTION 59 (VIPUL NAIK)

1. Performance review

27 people took this 5-question quiz. The score distribution was as follows:

- Score of 1: 4 people (this was mostly people who missed class!)
- Score of 2: 7 people
- Score of 3: 12 people
- Score of 4: 2 people
- Score of 5: 2 people

The question wise answers and performance summary:

(1) Option (B): 21 people.
(2) Option (D): 8 people. *Please review this solution!*
(3) Option (A): 12 people. *Please review this solution!*
(4) Option (C): 20 people.
(5) Option (D): 11 people. *Please review this solution!*

2. Solutions

(1) Suppose f is a function of two variables, defined on all of $\mathbb{R}^2$, with the property that $f(x, y) = f(y, x)$ for all real numbers x and y. What does this say about the symmetry of the graph $z = f(x, y)$ of f?
 (A) It has mirror symmetry about the plane $z = x + y$.
 (B) It has mirror symmetry about the plane $x = y$.
 (C) It has mirror symmetry about the plane $z = x - y$.
 (D) It has half turn symmetry about the line $x = y = z$.
 (E) It has half turn symmetry about the origin.

 Answer: Option (B)

 Explanation: The condition $f(x, y) = f(y, x)$ implies that if the point (x, y, z) lies in the graph, so does the point (y, x, z). These two points are mirror images of each other with respect to the plane $x = y$.

 Performance review: 21 out of 27 got this. 4 chose (D), 1 each chose (A) and (E).

 Historical note (last time): 16 out of 21 people got this correct. 2 each chose (A) and (D) and 1 chose (E).

(2) Consider the function $f(x, y) := ax + by$ where a and b are fixed nonzero reals. The level curves for this function are a bunch of parallel lines. What vector are they all parallel to?
 (A) $\langle a, b\rangle$
 (B) $\langle a, -b\rangle$.
 (C) $\langle b, a\rangle$
 (D) $\langle b, -a\rangle$
 (E) $\langle a - b, a + b\rangle$

 Answer: Option (D)

 Explanation: This can be seen by noting that the slope of the line $ax + by = c$ is $-a/b$. It can also be seen using dot products. The expression $ax + by$ is the dot product of the vector $\langle a, b\rangle$ and the vector $\langle x, y\rangle$. To keep this dot product constant (i.e., move along a level curve) one must move along a vector orthogonal to $\langle a, b\rangle$. Of the given vectors, $\langle b, -a\rangle$ is orthogonal to $\langle a, b\rangle$.

 Note that it is true that all the lines are *perpendicular* to $\langle a, b\rangle$, but they are not parallel to $\langle a, b\rangle$.

Performance review: 8 out of 27 got this. 12 chose (B), 6 chose (A), 1 chose (E).

Historical note (last time): 5 out of 21 got the question correct. 14 chose (A), 2 chose (B).

(3) Suppose f is a function of one variable and g is a function of two variables. What is the relationship between the level curves of $f \circ g$ and the level curves of g?

(A) Each level curve of $f \circ g$ is a union of level curves of g corresponding to the pre-images of the point under f.

(B) Each level curve of $f \circ g$ is an intersection of level curves of g corresponding to the pre-images of the point under f.

(C) The level curves of $f \circ g$ are precisely the same as the level curves of g.

(D) Each level curve of g is a union of level curves of $f \circ g$.

(E) Each level curve of g is an intersection of level curves of $f \circ g$.

Answer: Option (A)

Explanation: If $(f \circ g)(x, y) = c$, this means that $f(g(x, y)) = c$, so $g(x, y)$ is one of the pre-images of c under f. The set of possibilities for (x, y) is thus the union of the set of level curves for each of the pre-images of c under f.

Basically, the application of f can unite level curves, but it cannot separate them again, because once the g-values already agree, the $f \circ g$-values must also agree.

Performance review: 12 out of 27 got this. 6 chose (E), 5 chose (B), 3 chose (D), 1 chose (C).

Historical note (last time): 7 out of 21 people got this correct. 8 chose (B), 3 chose (C), 3 chose (E).

(4) Consider the following function f from $\mathbb{R}^2$ to $\mathbb{R}^2$: the function that sends $\langle x, y \rangle$ to $\langle \frac{x+y}{2}, \frac{x-y}{2} \rangle$. What is the image of $\langle x, y \rangle$ under $f \circ f$?

(A) $\langle x, y \rangle$

(B) $\langle 2x, 2y \rangle$

(C) $\langle x/2, y/2 \rangle$

(D) $\langle x + (y/2), y + (x/2) \rangle$

(E) $\langle 2x + y, 2x - y \rangle$

Answer: Option (C)

Explanation: We apply f to $\langle (x + y)/2, (x - y)/2 \rangle$ and get the first coordinate as $((x + y)/2 + (x - y)/2)/2 = x/2$ and the second coordinate as $((x + y)/2 - (x - y)/2)/2 = y/2$.

Performance review: 20 out of 27 got this. 3 chose (A), 2 chose (D), 1 each chose (B) and (E).

Historical note (last time): 12 out of 21 people got this correct. 4 chose (A), 3 chose (D), 2 chose (E).

(5) Consider the following functions defined on the subset $x > 0$ of the xy-plane: $f(x, y) = x^y$. Consider the surface $z = f(x, y)$. What do the intersections of this surface with planes parallel to the xz-plane and yz-plane look like (ignore the following two special intersections: intersection with the plane $x = 1$ and intersection with the plane $y = 0$, also ignore intersections that turn out to be empty).

(A) Intersections with any plane parallel to the xz or yz plane look like graphs of exponential functions.

(B) Intersections with any plane parallel to the xz or yz plane look like graphs of power functions (only positive inputs allowed).

(C) Intersections with any plane parallel to the xz-plane look like graphs of exponential functions, and intersections with any plane parallel to the yz-plane look like graphs of power functions (only positive inputs allowed).

(D) Intersections with any plane parallel to the yz-plane look like graphs of exponential functions, and intersections with any plane parallel to the xz-plane look like graphs of power functions (only positive inputs allowed).

(E) All the intersections are straight lines.

Answer: Option (D)

Explanation: A plane parallel to the yz-plane corresponds to fixing a value of x. The intersection with such a plane is the graph of the function $y \mapsto x^y$ with x a constant. By assumption, $x \neq 1$ and

$x > 0$, so if we set $k = \ln x$, this becomes $y \mapsto \exp(ky)$. This is an exponential function (increasing if $k > 0$, decreasing if $k < 0$).

A plane parallel to the xz-plane corresponds to a fixed value of x. The intersection with such a plane is the graph of the function $x \mapsto x^y$ with y a constant. By assumption $y \neq 0$. We thus get a power function, and x is restricted to being positive.

Performance review: 11 out of 27 got this. 8 chose (C), 6 chose (E), 2 chose (A).

Historical note (last time): 5 out of 21 people got this correct. 8 chose (C), 3 each chose (B) and (E), and 2 chose (A).

TAKE-HOME CLASS QUIZ SOLUTIONS: DUE WEDNESDAY FEBRUARY 6: MULTIVARIABLE FUNCTION BASICS CONTINUED

MATH 195, SECTION 59 (VIPUL NAIK)

1. Performance review

26 people took this quiz. The score distribution was as follows:

- Score of 0: 4 people
- Score of 1: 6 people
- Score of 2: 9 people
- Score of 3: 5 people
- Score of 4: 2 people

The question-wise answers and performance review were as follows:

(1) Option (C): 12 people
(2) Option (B): 13 people
(3) Option (E): 15 people
(4) Option (B): 7 people

2. Solutions

(1) Suppose F is an additively separable function of two variables x and y that is defined everywhere, i.e., there exist functions f and g of one variable, both defined on all of $\mathbb{R}$, such that $F(x, y) = f(x) + g(y)$ for all $x, y \in \mathbb{R}$.

We call two curves *parallel* if there is a vector by which we can translate all the points in one curve to get precisely the other curve.

Consider the following three statements:

(i) All curves obtained as the intersections of the graph of F with planes parallel to the xy-plane are parallel to each other.

(ii) All curves obtained as the intersections of the graph of F with planes parallel to the xz-plane are parallel to each other.

(iii) All curves obtained as the intersections of the graph of F with planes parallel to the yz-plane are parallel to each other.

Which of the statements (i)-(iii) is/are necessarily true?

(A) All of (i), (ii), and (iii) are true.
(B) Both (i) and (ii) are true but (iii) need not be true.
(C) Both (ii) and (iii) are true but (i) need not be true.
(D) Both (i) and (iii) are true but (ii) need not be true.
(E) (i) is true but (ii) and (iii) need not be true.

Answer: Option (C)

Explanation:

(i): The intersections are level curves, but these need not be parallel to each other. In fact, they could be different sizes, such as with $x^2 + y^2$.

(ii): The intersection with a plane of the form $y = y_0$ gives the graph of a function $x \mapsto F(x, y_0) = f(x) + g(y_0)$. Note that all the functions whose graphs are obtained by such restrictions just look like the graph of f, translated to different z-heights and y-locations. Thus, they are parallel to one another. Note that we use additive separability in this reasoning.

(iii): Similar reasoning as with (ii).

Performance review: 12 out of 26 got this. 7 chose (A), 6 chose (E), 1 chose (B).

(2) Suppose f is a continuous function of two variables x and y, defined on the entire xy-plane. Suppose further that f is increasing in x for each fixed value of y, and that f is increasing in y for every fixed value of x. Which of the following is the most plausible description of the level curves of f in the xy-plane? *Note: You might wish to take an extremely simple example, e.g., an additively separable function where each of the pieces is the simplest possible increasing function you can think of.*

(A) They are all upward-sloping, i.e., they are of the form $y = g(x)$ with g an increasing function.
(B) They are all downward-sloping, i.e., they are of the form $y = g(x)$ with g a decreasing function.
(C) They look like closed loops (e.g., circles).
(D) They look like graphs of functions with a unique local and absolute minimum (such as the parabola $y = x^2$, though the actual picture may be different).
(E) They look like graphs of functions with a unique local and absolute maximum (such as the parabola $y = -x^2$, though the actual function may be different).

Answer: Option (B)

Explanation: Since f is increasing in x and in y, it means that an increase in the x-value must be compensated by a decrease in the y-value to keep the output constant.

The example $f(x, y) := x + y$ is illustrative. The level curves of these are downward-sloping straight lines.

Performance review: 13 out of 26 got this. 7 chose (A), 3 each chose (C) and (D).

(3) What do the level curves of the function $f(x, y) := \sin(x + y)$ look like for output value in $[-1, 1]$? Note that all these level curves are being considered as curves in the xy-plane. *Note: This builds upon the idea of Question 3 of the previous quiz.*

(A) Each level curve is a single line.
(B) Each level curve is a union of two intersecting lines.
(C) Each level curve is a union of two distinct parallel lines.
(D) Each level curve is a union of infinitely many concurrent lines (i.e., infinitely many lines, all passing through the same point).
(E) Each level curve is a union of infinitely many distinct parallel lines (i.e., infinitely many lines, all parallel to each other).

Answer: Option (E)

Explanation: For $C \in [-1, 1]$, we need to solve $\sin(x + y) = C$. We first find all solutions u to $\sin u = C$, which is a countably infinite subset of $\mathbb{R}$. Then, for each such u, we get the line $x + y = u$ in the xy-plane. All these lines are parallel to each other, and have slope -1.

Performance review: 15 out of 26 got this. 6 chose (A), 4 chose (D), 1 chose (B).

(4) Suppose f and g are both continuous functions of two variables x and y, both defined on all of $\mathbb{R}^2$, and such that $f(x, y) + g(x, y)$ is a constant C. What is the relation between the level curves of f and the level curves of g, all drawn in the xy-plane?

(A) Every level curve of f is a level curve of g and vice versa, with the same level value for both functions.
(B) Every level curve of f is a level curve of g and vice versa, but the value for which it is a level curve may be different for the two functions.
(C) The level curves of f need not be precisely the same as the level curves of g, but we can go from one set of level curves to the other via a parallel translation.
(D) Each level curve of f can be obtained by reflecting a suitable level curve of g about a suitable line in the xy-plane.
(E) Each level curve of f can be obtained by reflecting a suitable level curves of g about a suitable line in the xy-plane and then performing a suitable translation.

Answer: Option (B)

Explanation: Any level curve of the form $f(x, y) = k$ coincides with the level curve $g(x, y) = C - k$. Note that unless $k = C/2$, the level values for the two curves are different.

Performance review: 7 out of 26 got this. 8 chose (D), 6 chose (C), 3 chose (A), 2 chose (E).

CLASS QUIZ SOLUTIONS: WEDNESDAY FEBRUARY 6: MULTIVARIABLE LIMIT COMPUTATIONS

MATH 195, SECTION 59 (VIPUL NAIK)

1. Performance review

26 people took this quiz. The score distribution was as follows:

- Score of 0: 1 person.
- Score of 1: 2 people.
- Score of 2: 9 people.
- Score of 3: 10 people.
- Score of 4: 4 people.

The question-wise answers and performance review were as follows:

(1) Option (A): 18 people
(2) Option (D): 24 people
(3) Option (B): 9 people
(4) Option (B): 15 people

2. Solutions

(1) Consider the function $f(x, y) := x \sin(1/(x^2 + y^2))$, defined on all points other than the point $(0, 0)$. What is the limit of the function at $(0, 0)$?
   (A) 0
   (B) $1/\sqrt{2}$
   (C) 1
   (D) The limit is undefined, because the expression becomes unbounded around 0.
   (E) The limit is undefined, because the expression is oscillatory around 0.

   Answer: Option (A)

   Explanation: We know that $\sin(1/(x^2 + y^2))$ is bounded in $[-1, 1]$, and $x \to 0$ as we approach the origin. We thus get the product of something approaching 0 and something bounded, which is therefore 0.

   Performance review: 18 out of 26 got this. 7 chose (E), 1 chose (D).

   Historical note (last time): 8 out of 22 people got this correct. 12 chose (E). 1 each chose (C) and (D).

   Some failed to note the x on the outside. Others thought that the behavior of $\sin(1/(x^2 + y^2))$ is like the behavior of $1/(x^2 + y^2)$ near the origin. This is not true. Near the origin, $1/(x^2 + y^2)$ approaches ∞ but sin of the same quantity is oscillatory. Note also that sin cannot be stripped off because the input to it is not going to 0.

(2) The typical $\varepsilon - \delta$ definition of limit in two dimensions makes use of open disks centered at the points on the domain and range side, where the open disk is the interior region bounded by a circle centered at the point. Which other geometric shapes can we use instead of a circle of specified radius centered at the point?
   (A) A square of specified side length centered at the point
   (B) An equilateral triangle of specified side length centered at the point
   (C) A regular hexagon of specified side length centered at the point
   (D) Any of the above
   (E) None of the above

Answer: Option (D)

Explanation: Basically, any shape that is bounded both from inside and from outside by a circle will do.

Performance review: 24 out of 26 got this. 1 each chose (B) and (E).

Historical note (last time): 12 out of 22 people got this correct. 6 chose (A), 4 chose (E).

(3) Here's a quick recap of the limit definition for a function of a vector variable. We say that $\lim_{\mathbf{x}\to\mathbf{c}} f(\mathbf{x}) = L$ if for every $\varepsilon > 0$ there exists $\delta > 0$ such that for all $\mathbf{x}$ satisfying $0 < |\mathbf{x} - \mathbf{c}| < \delta$, we have $|f(\mathbf{x}) - L| < \varepsilon$. We define $|\mathbf{x} - \mathbf{c}|$ as the Euclidean norm of $\mathbf{x} - \mathbf{c}$ where the Euclidean norm of a vector is the square root of the sum of the squares of its coordinates.

We could replace the Euclidean norm by other measurements. For instance, we could use:

(i) The *sum* of the absolute values of the coordinates of $\mathbf{x} - \mathbf{c}$.

(ii) The *maximum* of the absolute values of the coordinates of $\mathbf{x} - \mathbf{c}$.

(iii) The *minimum* of the absolute values of the coordinates of $\mathbf{x} - \mathbf{c}$.

For any of (i) - (iii), we could replace $|\mathbf{x} - \mathbf{c}|$ in our current definition of limit with that notion. The question is: for which of the replacements will our new notion of limit be the same as the old one? The deeper idea here is that limit depends upon a concept of what it means for two points to be close. So another way of phrasing the question is: which of the notions (i)-(iii) capture the same notion of closeness as the usual Euclidean distance?

(A) All of (i), (ii), and (iii).
(B) (i) and (ii) but not (iii).
(C) (i) and (iii) but not (ii).
(D) Only (i).
(E) None of (i), (ii), or (iii).

Answer: Option (B)

Explanation: The sum of the absolute values of the coordinates of a vector is small if and only if all the absolute values of the coordinates are small. Similarly, the maximum of the absolute values of the coordinates is small if and only if all the coordinates are small. On the other hand, the minimum could be small even if some of the coordinates are very large. For this reason, the minimum does not capture the correct notion of "closeness" – in the notion of closeness it gives, very far-away points can appear close merely because they are close in one coordinate.

Performance review: 9 out of 26 people got this. 16 chose (C), 1 chose (D).

(4) Suppose f is a function of two variables x, y and is defined on the whole xy-plane. Consider three conditions: (i) f is continuous on the whole xy-plane, (ii) for every fixed value $x = x_0$, the function $y \mapsto f(x_0, y)$ is continuous in y for all $y \in \mathbb{R}$, (iii) for every fixed value $y = y_0$, the function $x \mapsto f(x, y_0)$ is continuous in x for all $x \in \mathbb{R}$, (iv) the function $t \mapsto f(p(t), q(t))$ is continuous for all $t \in \mathbb{R}$ whenever p and q are both constant or linear functions (in other words, the restriction of f to any straight line in $\mathbb{R}^2$ is continuous).

Which of the following correctly describes the implications between (i), (ii), (iii), and (iv)?

(A) (i) implies both (ii) and (iii), and (ii) and (iii) together imply (iv).
(B) (i) implies (iv), and (iv) implies both (ii) and (iii).
(C) (iv) implies (ii) and (iii), and (ii) and (iii) together imply (i).
(D) (iv) implies (i), and (i) implies both (ii) and (iii).
(E) (ii) and (iii) together imply (iv), and (iv) implies (i).

Answer: Option (B)

Explanation: Continuous implies continuous in every direction, linear and curved, hence (i) implies (iv) (we can also think of this as a result of composition of continuous functions being continuous). (iv) implies (ii) and (iii) because (ii) and (iii) are continuous from specific linear directions, namely the directions parallel to the coordinate axes.

That the reverse implicationa fail is covered in the lecture notes and in videos.

Performance review: 15 out of 26 got this. 4 chose (A), 3 each chose (D) and (E). 1 left the question blank.

TAKE-HOME CLASS QUIZ SOLUTIONS: DUE MONDAY FEBRUARY 18: PARTIAL DERIVATIVES

MATH 195, SECTION 59 (VIPUL NAIK)

1. Performance review

26 people took this quiz. The score distribution was as follows:

- Score of 3: 1 person.
- Score of 4: 1 person.
- Score of 5: 2 people.
- Score of 6: 1 person.
- Score of 7: 2 people.
- Score of 8: 3 people.
- Score of 9: 5 people.
- Score of 11: 5 people.
- Score of 12: 3 people.
- Score of 13: 3 people.

The question wise answers and performance review were as follows:

(1) Option (D): 17 people.
(2) Option (A): 3 people.
(3) Option (A): 15 people.
(4) Option (E): 16 people.
(5) Option (A): 18 people.
(6) Option (E): 4 people.
(7) Option (C): 2 people.
(8) Option (B): 0 people.
(9) Option (C): 8 people.
(10) Option (C): 20 people.
(11) Option (E): 19 people.
(12) Option (D): 21 people.
(13) Option (B): 24 people.
(14) Option (D): 25 people.
(15) Option (B): 14 people.
(16) Option (D): 15 people.
(17) Option (E): 15 people.

2. Solutions

(1) For this and the next question, consider the function on $\mathbb{R}^2$ given as:

$$f(x, y) := \begin{cases} 1, & x \text{ rational or } y \text{ rational} \\ 0, & x \text{ and } y \text{ both irrational} \end{cases}$$

What can we say about the subset S of $\mathbb{R}^2$ defined as the set of points where f_x is defined?

(A) S is the set of points for which at least one coordinate is rational.
(B) S is the set of points for which both coordinates are rational.
(C) S is the set of points for which the x-coordinate is rational.
(D) S is the set of points for which the y-coordinate is rational.
(E) S is the set of points for which at least one coordinate is irrational.

Answer: Option (D)

Explanation: f_x means we take the derivative with respect to x holding y constant. Consider the point (x_0, y_0). If y_0 is rational, then $f(x, y_0) = 1$ on the entire line $y = y_0$. Thus, $f_x(x_0, y_0)$ is the derivative of a constant function, hence is 0. In particular, it is well defined.

On the other hand, if y_0 is irrational, then there are points (x, y_0) for x arbitrarily close to x_0 for which x is rational, giving $f(x, y_0) = 1$, and also points where x is irrational, giving $f(x, y_0) = 0$. Thus, f is not continuous in x at (x_0, y_0), and hence $f_x(x_0, y_0)$ does not exist.

Performance review: 17 out of 26 people got this. 4 chose (A), 2 each chose (C) and (E), 1 chose (B).

(2) With f as in the previous question, what is the subset T of $\mathbb{R}^2$ at which the second-order mixed partial derivative f_{xy} is defined?
(A) T is the empty subset.
(B) T is the set of points for which both coordinates are rational.
(C) T is the set of points for which the x-coordinate is rational.
(D) T is the set of points for which the y-coordinate is rational.
(E) T is the set of points for which both coordinates are irrational.

Answer: Option (A)

Explanation: We have $f_{xy} = (f_x)_y$. In order for this to be defined at (x_0, y_0), a necessary condition is that f_x be defined at (x_0, y_0), and also that it be defined at (x_0, y) for y close to y_0. Note that the former condition holds only if y_0 is rational. However, the latter condition is never true, because for any value of y_0, there are values y arbitrarily close to y_0 that are rational, and also values y that are irrational.

Performance review: 3 out of 26 people got this. 19 chose (B), 3 chose (D), 1 chose (C).

(3) For this and the next three questions, consider the function on $\mathbb{R}^2$ given as:

$$g(x, y) := \begin{cases} 1, & x \text{ rational} \\ 0, & x \text{ irrational} \end{cases}$$

What can we say about the subset U of $\mathbb{R}^2$ defined as the set of points where g_x is defined?
(A) U is the empty subset.
(B) U is the set of points for which both coordinates are rational.
(C) U is the set of points for which the x-coordinate is rational.
(D) U is the set of points for which the y-coordinate is rational.
(E) U is the whole plane $\mathbb{R}^2$.

Answer: Option (A)

Explanation: At any point (x_0, y_0), there are x-values arbitrarily close to x_0 that are rational, and x-values arbitrarily close to x_0 that are irrational. Thus, g is not continuous in x at any point, so g_x does not exist anywhere.

Performance review: 15 out of 26 got this. 10 chose (C), 1 chose (E).

(4) With g as in the preceding question, what can we say about the subset V of $\mathbb{R}^2$ defined as the set of points where g_y is defined?
(A) V is the empty subset.
(B) V is the set of points for which both coordinates are rational.
(C) V is the set of points for which the x-coordinate is rational.
(D) V is the set of points for which the y-coordinate is rational.
(E) V is the whole plane $\mathbb{R}^2$.

Answer: Option (E)

Explanation: Note that g depends *only* on x, hence it is independent of y. Thus, g_y is identically the zero function, and is defined everywhere.

Performance review: 16 out of 26 got this. 5 chose (D), 2 each chose (A) and (C), 1 chose (B).

(5) With g as in the preceding question, what can we say about the subset W of $\mathbb{R}^2$ defined as the set of points where g_{xy} is defined?
(A) W is the empty subset.
(B) W is the set of points for which both coordinates are rational.

(C) W is the set of points for which the x-coordinate is rational.
(D) W is the set of points for which the y-coordinate is rational.
(E) W is the whole plane $\mathbb{R}^2$.

Answer: Option (A)

Explanation: This follows from g_x not being defined anywhere.

Performance review: 18 out of 26 got this. 4 chose (B), 2 chose (E), 1 each chose (C) and (D).

(6) With g as in the preceding question, what can we say about the subset X of $\mathbb{R}^2$ defined as the set of points where g_{yx} is defined?
(A) X is the empty subset.
(B) X is the set of points for which both coordinates are rational.
(C) X is the set of points for which the x-coordinate is rational.
(D) X is the set of points for which the y-coordinate is rational.
(E) X is the whole plane $\mathbb{R}^2$.

Answer: Option (E)

Explanation: This follows from g_y being the zero function everywhere.

Performance review: 18 out of 26 got this. 4 chose (E), 3 chose (B), 1 chose (C).

(7) For this and the next two questions, consider the function on $\mathbb{R}^2$ given as:

$$h(x,y) := \begin{cases} 1, & x \text{ an integer or } y \text{ an integer} \\ 0, & x \text{ not an integer and } y \text{ not an integer} \end{cases}$$

What can we say about the subset A of $\mathbb{R}^2$ defined as the set of points where h_{xy} is defined?
(A) A is the empty set.
(B) A is the set of points whose x-coordinate is an integer.
(C) A is the set of points whose x-coordinate is not an integer.
(D) A is the set of points whose y-coordinate is an integer.
(E) A is the set of points whose y-coordinate is not an integer.

Answer: Option (C)

Explanation: We first note that if both coordinates are non-integers, then h is identiically the zero function *at and around* the point, so all first and higher order partials are zero. In particular, $h_x = h_{xy} = 0$ at all points with both coordinate non-integers.

Suppose now that the y-coordinate is an integer. In this case, h is identically 1 on lines of the form $y = y_0$, y_0 an integer. Thus, h_x is zero on these lines.

Suppose now that the x-coordinate is an integer but the y-coordinate is a non-integer. In this case, the function h takes the value 1 at the point, but takes the value 0 if we vary the x-coordinate even slightly. Thus, h_x is not defined at such points.

Thus, overall, h_x is defined and equal to zero at all points where either the y-coordinate is an integer or both coordinates are non-integers. It is not defined precisely at the points where the x-coordinate is an integer and the y-coordinate is a non-integer.

Of the points where h_x is defined, h_{xy} is not defined at points where both coordinates are integers, because slightly perturbing the y-coordinate gets a point where h_x is undefined. h_{xy} is defined and equal to zero at all other points, namely the points where the x-coordinate is a non-integer.

Performance review: 2 out of 26 got this. 11 chose (D), 8 chose (A), 4 chose (B), 1 left the question blank.

(8) With h as defined in the previous question, what can we say about the subset B of $\mathbb{R}^2$ defined as the set of points where h_x is defined but h_{xy} is not defined?
(A) B is the empty set.
(B) B is the set of points for which both coordinates are integers.
(C) B is the set of points for which both coordinates are non-integers.
(D) B is the set of points for which at least one coordinate is an integer.
(E) B is the set of points for which at least one coordinate is a non-integer.

Answer: Option (B)

Explanation: See the explanation for the preceding question.

Performance review: Nobody got this correct. 8 chose (D), 7 chose (E), 5 each chose (A) and (C). 1 left the question blank.

(9) With h as defined in the previous question, what can we say about the subset C of $\mathbb{R}^2$ defined as the set of points where both h_{xy} and h_{yx} are defined?

(A) C is the empty set.
(B) C is the set of points for which both coordinates are integers.
(C) C is the set of points for which both coordinates are non-integers.
(D) C is the set of points for which at least one coordinate is an integer.
(E) C is the set of points for which at least one coordinate is a non-integer.

Answer: Option (C)

Explanation: By the question before last, the set of points where h_{xy} is defined is the set of points where the x-coordinate is a non-integer. The function is symmetric in x and y, so analogous reasoning yields that the set of points where hyx is defined is the set of points where the y-coordinate is a non-integer. Intersecting the two sets, we get the set of points where both coordinates are non-integers.

Performance review: 8 out of 26 got this correct. 14 chose (B), 2 chose (A), 1 chose (D), 1 left the question blank.

(10) Students training for an examination can spend money either on purchasing textbooks or on private tuitions. A student's expected performance on the examination is a function of the money the student spends on textbooks and on tuition (viewed as separate variables). Two researchers want to consider the question of whether increased expenditure on textbooks leads to improved performance on the examination, and if so, by how much.

One researcher decides to measure the increase in the examination score for a marginal increase in textbook expenditure *holding constant the expenditure on tuitions*, arguing that in order to determine the effect of changes in textbook expenditures, the other expenditures need to be kept constant.

The other researcher believes that since the student has a limited budget, it would be more realistic to measure the increase in the examination score for a marginal increase in textbook expenditure *holding constant the total expenditure on both textbook and tuitions*. This is because the student is likely to allocate money away from tuition expenditures in order to spend money on textbooks.

Which of the following best describes what's happening?

(A) Both researchers are effectively computing the same quantity.
(B) The two quantities that the researchers are computing have a simple linear relationship, i.e., their sum or difference is a constant.
(C) The two quantities that the researchers are computing are meaningfully different and there is a relationship between them but that relationship involves other partial derivatives.

Answer: Option (C)

Explanation: Too tricky to review here, but you might want to watch this video and subsequent ones in the playlist:

`http://www.youtube.com/watch?v=tfH2iqt2E0E&list=PLC0bHnWu122kC1WBgr0H9PEbHTYnYev27&index=4`

Performance review: 20 out of 26 got this correct. 3 each chose (A) and (B).

(11) F is an everywhere twice differentiable function of two variables x and y. Which of the following captures the manner in which the inputs x and y *interact* with each other in the description of F?

(A) The difference $F_x - F_y$
(B) The quotient F_x/F_y.
(C) The product F_xF_y.
(D) The product $F_{xx}F_{yy}$.
(E) The mixed partial F_{xy}

Answer: Option (E)

Explanation: One extreme way of seeing this is that if F is additively separable (i.e., it is the sum of a function of x and a function of y) then $F_{xy}(x, y) = 0$. Thus, the stereotypical case in which the variables don't interact with each other is the case that the second-order mixed partial is zero.

Performance review: 19 out of 26 got this correct. 4 chose (D), 2 chose (B), 1 chose (C).

(12) F is a function of two variables x and y such that both F_x and F_y exist. Which of the following is generically true?

(A) In general, F_x depends only on x (i.e., it is independent of y) and F_y depends only on y. An exception is if F is multiplicatively separable.

(B) In general, F_x depends only on y (i.e., it is independent of x) and F_y depends only on x (i.e., it is independent of y). An exception is if F is multiplicatively separable.

(C) In general, both F_x and F_y could each depend on both x and y. An exception is if F is additively separable, in which case F_x depends only on y and F_y depends only on x.

(D) In general, both F_x and F_y could each depend on both x and y. An exception is if F is additively separable, in which case F_x depends only on x and F_y depends only on y.

(E) In general, either both F_x and F_y depend only on x or both F_x and F_y depend only on y.

Answer: Option (D)

Explanation: This should be straightforward if you understand what's going on. Otherwise, watch this video and the subsequent one:

`http://www.youtube.com/watch?v=2T7iFZVLtn0&list=PLC0bHnWu122kC1WBgr0H9PEbHTYnYev27&index=1`

Performance review: 21 out of 26 got this correct. 3 chose (A), 1 each chose (C) and (E).

(13) Consider a production function $f(L, K, T)$ of three inputs L (labor expenditure), K (capital expenditure), and T (technology expenditure). Suppose all mixed partials of f with respect to L, K, and T are continuous. Suppose we have the following signs of partial derivatives: $\partial f/\partial L > 0$, $\partial f/\partial K > 0$, $\partial^2 f/(\partial L \partial K) < 0$, and $\partial^3 f/(\partial L \partial K \partial T) > 0$. What does this mean?

(A) Increasing labor increases production, increasing capital increases production, and labor and capital substitute for each other to some extent. Increasing the expenditure on technology increases the degree to which labor and capital substitute for each other.

(B) Increasing labor increases production, increasing capital increases production, and labor and capital substitute for each other to some extent. Increasing the expenditure on technology decreases the degree to which labor and capital substitute for each other, i.e., with more technology investment, labor and capital become more complementary.

(C) Increasing labor increases production, increasing capital increases production, and labor and capital complement each other to some extent. Increasing the expenditure on technology increases the degree to which labor and capital complement for each other.

(D) Increasing labor increases production, increasing capital increases production, and labor and capital complement each other to some extent. Increasing the expenditure on technology decreases the degree to which labor and capital complement for each other.

(E) Increasing labor or capital decreases production.

Answer: Option (B)

Explanation: $\partial f/\partial L > 0$ shows that increasing labor increases production. $\partial f/\partial K > 0$ shows that increasing capital increases production. $\partial^2 f/\partial L \partial K < 0$ indicates that labor and capital substitute for each other, i.e., a small increase in capital reduces the marginal product of labor. Finally, $\partial^3 f/\partial L \partial K \partial T > 0$ indicates that $\partial^2 f/\partial L \partial K$ is increasing with T, i.e., getting less negative. So, although labor and capital substitute for each other, the degree to which they do so reduces as T increases. Roughly speaking, more technology reduces the antagonism between labor and capital.

Performance review: 24 out of 26 got this correct. 1 each chose (A) and (C).

Historical note (last time): 12 out of 20 people got this correct. 7 chose (A) and 1 chose (D). The people who chose (A) probably didn't note that an increase in the degree of substitution would mean a decrease in the derivative, rather than an increase.

(14) Analysis of usage of an online social network finds that the total time spent by people on the social network is $P^{1.3}L^{0.5}$ where P is the total number of people on the network and L is a number of processors used at the social network's server facility. Which of these is true?

(A) Increasing returns both on persons and on processors: every new person joining the network increases the average time spent *per person* (and not just the total time), and every new processor added to the server facility increases the average time spent per processor.

(B) Constant returns on persons, increasing returns on processors

(C) Constant returns on persons, decreasing returns on processors
(D) Increasing returns on persons, decreasing returns on processors
(E) Decreasing returns on persons, increasing returns on processors

Answer: Option (D)

Explanation: The short explanation is that the exponent on P is greater than 1, so the second partial derivative is positive, and the exponent of L is between 0 and 1, so the second partial derivative is negative.

The long explanation is just working it out.

Performance review: 25 out of 26 got this. 1 chose (C).

Historical note (last time): 14 out of 20 got this correct. 5 chose (A), 1 chose (B).

(15) *Not a calculus question, but has deep calculus interpretations – it is basically measuring the derivative of the* $1/x$ *function with respect to* x: A person travels fifty miles every day by car and the travel distance is fixed. The price of gasoline, which she uses to fuel her car, is also fixed. Which of the following increases in fuel efficiency result in the maximum amount of savings for her?
(A) From 11 to 12 miles per gallon
(B) From 12 to 14 miles per gallon
(C) From 20 to 25 miles per gallon
(D) From 36 to 54 miles per gallon
(E) From 50 to 100 miles per gallon

Answer: Option (B)

Explanation: The gain achieved by upgrading from a miles per gallon to b miles per gallon is $(50/a - 50/b)$ times the cost of a gallon. In particular, it is proportional to $1/a - 1/b$. It remains to compute the case where this difference is largest.

The values of $1/a - 1/b$ are: Option (A):/ 1/132, Option (B): 1/84, Option (C): 1/100, Option (D): 1/108, Option (E): 1/100. Of these, the largest is the one with smallest denominator, i.e., 1/84. In other words, the maximum gain happens in going from 12 to 14.

This seems a little counter-intuitive at first. Looked at in terms of ratios, the gain from 50 to 100 is most impressive. Looked at in terms of differences in MPG values, again the gain from 50 to 100 is more impressive. However, these gains are not what we are measuring, because in the question, it is specified that the distance of travel is *fixed* and hence what matters is the absolute savings in cost.

Intuitively, what's happening is that while a gain from 50 to 100 halves the cost, that halving is occurring from an already fairly small cost base, so the quantitative savings are little. On the other hand, a jump from 12 to 14 is small in proportion but large in absolute terms because the base from which the savings are occurring is much larger. In fact, even a gain from 100 miles per gallon to infinite miles per gallon produces less in cost savings assuming fixed distance and fixed cost per gallon than a gain from 12 to 14.

Another way of thinking of this is in terms of the derivative of $1/x$. We know that as x increases, $1/x$ decreases. However, the derivative is not constant. When x is small, the derivative $-1/x^2$ is huge in magnitude, which means that small changes in x lead to large changes in $1/x$. When x is large, the derivative $-1/x^2$ is small in magnitude, which means that large changes in x lead to only small changes in $1/x$.

Thus, we can get three fairly different pictures depending on whether we measure things using x, $1/x$, or $\ln x$.

Performance review: 14 out of 26 people got this. 6 chose (E), 3 chose (A), 2 chose (D), 1 chose (C).

Historical note (last time): 2 out of 20 people got this correct. 7 chose (C), 6 chose (E), 4 chose (D), and 1 chose (A).

(16) For which of the following production functions $f(L, K)$ of labor and capital is it true that labor and capital can be complementary for some choices of (L, K), and substitutes for others? In other words, for which of these are labor and capital neither globally complements nor globally substitutes? Assume the domain $L > 0, K > 0$.
(A) $L^2 + LK + K^2$

(B) $L^2 - LK + K^2$
(C) $L^3 + L^2K + LK^2 + K^3$
(D) $L^3 + L^2K - LK^2 + K^3$
(E) $L^3 - L^2K - LK^2 + K^3$

Answer: Option (D)

Explanation: For option (D), the second mixed partial is $2(L - K)$ which is positive if $L > K$ and negative if $L < K$. For options (A) and (C), the second mixed partial is always positive, while for options (B) and (E), the second mixed partial is always negative.

Performance review: 15 out of 26 got this. 6 chose (E), 3 chose (B), 2 chose (C).

Historical note (last time): 8 out of 20 people got this correct. 4 each chose (B), (C) and (E).

(17) Consider the following Leontief-like production function $f(L, K) = (\min\{L, K\})^2$. Assume the domain $L > 0$, $K > 0$. What is the nature of returns and complementarity here?

(A) Positive increasing returns on the smaller of the inputs, positive constant returns on the larger of the inputs
(B) Positive constant returns of the smaller of the inputs, positive increasing returns on the larger of the inputs
(C) Zero returns on the smaller of the inputs, positive constant returns on the larger of the inputs
(D) Positive decreasing returns on the smaller of the inputs, zero returns on the larger of the inputs
(E) Positive increasing returns on the smaller of the inputs, zero returns on the larger of the inputs

Answer: Option (E)

Explanation: This is a "weak link" type of production function in the sense that the weakest link in the labor-capital nexus determines output. If $L < K$, then output is L^2, and if $K < L$, then output is K^2. This means that, at the margin, increasing the one which is already larger produces no gain in output. However, increasing the one which is smaller increases output as the square thereof. Since $2 > 1$, there are positive increasing returns on the smaller input.

A practical example of this is where "it takes two to tango" – for instance, if each unit of labor is a person and each unit if capital is a machine, and if there are more machines than people or vice versa, the extra machines/people are completely unused.

Performance review: 15 out of 26 got this. 5 chose (A), 4 chose (D), 1 each chose (B) and (C).

Historical note (last time): 10 out of 20 people got this correct. 3 each chose (A) and (D), 2 each chose (B) and (C).

TAKE-HOME CLASS QUIZ SOLUTIONS: DUE WEDNESDAY FEBRUARY 20: INTEGRATION TECHNIQUES (ONE VARIABLE)

MATH 195, SECTION 59 (VIPUL NAIK)

1. Performance review

25 people took this quiz. The score distribution was as follows:

- Score of 3: 1 person.
- Score of 5: 1 person.
- Score of 6: 1 person.
- Score of 9: 1 person.
- Score of 10: 1 person.
- Score of 12: 1 person.
- Score of 15: 1 person.
- Score of 18: 7 people.
- Score of 19: 2 people.
- Score of 20: 4 people.
- Score of 21: 5 people.

Here are the question wise answers and performance review:

(1) Option (A): 20 people.
(2) Option (B): 22 people.
(3) Option (A): 17 people.
(4) Option (C): 19 people.
(5) Option (B): 19 people.
(6) Option (D): 21 people.
(7) Option (B): 20 people.
(8) Option (D): 22 people.
(9) Option (B): 21 people.
(10) Option (C): 21 people.
(11) Option (C): 20 people.
(12) Option (D): 20 people.
(13) Option (A): 22 people.
(14) Option (E): 21 people.
(15) Option (D): 22 people.
(16) Option (B): 15 people.
(17) Option (D): 21 people.
(18) Option (B): 17 people.
(19) Option (E): 9 people.
(20) Option (D): 23 people.
(21) Option (D): 17 people.

2. Solutions

In the questions below, we say that a function is *expressible in terms of elementary functions* or *elementarily expressible* if it can be expressed in terms of polynomial functions, rational functions, radicals, exponents, logarithms, trigonometric functions and inverse trigonometric functions using pointwise combinations, compositions, and piecewise definitions. We say that a function is *elementarily integrable* if it has an elementarily expressible antiderivative.

Note that if a function is elementarily expressible, so is its derivative on the domain of definition.

We say that a function f is k times elementarily integrable if there is an elementarily expressible function g such that f is the k^{th} derivative of g.

We say that the integrals of two functions are *equivalent up to elementary functions* if an antiderivative for one function can be expressed using an antiderivative for the other function and elementary function, again piecing them together using pointwise combination, composition, and piecewise definitions.

(1) Suppose F and G are continuously differentiable functions on all of $\mathbb{R}$ (i.e., both F' and G' are continuous). Which of the following is **not necessarily true**? Please see Option (E) before answering.
   (A) If $F'(x) = G'(x)$ for all integers x, then $F - G$ is a constant function when restricted to integers, i.e., it takes the same value at all integers.
   (B) If $F'(x) = G'(x)$ for all numbers x that are not integers, then $F - G$ is a constant function when restricted to the set of numbers x that are not integers.
   (C) If $F'(x) = G'(x)$ for all rational numbers x, then $F - G$ is a constant function when restricted to the set of rational numbers.
   (D) If $F'(x) = G'(x)$ for all irrational numbers x, then $F - G$ is a constant function when restricted to the set of irrational numbers.
   (E) None of the above, i.e., they are all necessarily true.
   Answer: Option (A).
   Explanation: The fact that the derivatives of two functions agree at integers says nothing about how the derivatives behave elsewhere – they could differ quite a bit at other places. Hence, (A) is not necessarily true, and hence must be the right option. All the other options are correct as statements and hence cannot be the right option. This is because in all of them, the set of points where the derivatives agree is *dense* – it intersects every open interval. So, continuity forces the functions F' and G' to be equal everywhere, forcing $F - G$ to be constant everywhere.
   Performance review: 20 out of 25 people got this. 3 chose (E), 1 each chose (B) and (D).
   Historical note (earlier appearance this quarter): 15 out of 24 got this. 6 chose (E), 3 chose (B).

(2) Suppose F and G are two functions defined on $\mathbb{R}$ and k is a natural number such that the k^{th} derivatives of F and G exist and are equal on all of $\mathbb{R}$. Then, $F - G$ must be a polynomial function. What is the **maximum possible degree** of $F - G$? (Note: Assume constant polynomials to have degree zero)
   (A) $k - 2$
   (B) $k - 1$
   (C) k
   (D) $k + 1$
   (E) There is no bound in terms of k.
   Answer: Option (B)
   Explanation: F and G having the same k^{th} derivative is equivalent to requiring that $F - G$ have k^{th} derivative equal to zero. For $k = 1$, this gives constant functions (polynomials of degree 0). Each time we increment k, the degree of the polynomial could potentially go up by 1. Thus, the answer is $k - 1$.
   Performance review: 22 out of 25 people got this. 2 chose (E), 1 chose (C).
   Historical note (earlier appearance this quarter): 21 out of 24 got this. 2 chose (A), 1 chose (E).

(3) Suppose f is a continuous function on $\mathbb{R}$. Clearly, f has antiderivatives on $\mathbb{R}$. For all but one of the following conditions, it is possible to guarantee, without any further information about f, that there exists an antiderivative F satisfying that condition. **Identify the exceptional condition** (i.e., the condition that it may not always be possible to satisfy).
   (A) $F(1) = F(0)$.
   (B) $F(1) + F(0) = 0$.
   (C) $F(1) + F(0) = 1$.
   (D) $F(1) = 2F(0)$.
   (E) $F(1)F(0) = 0$.
   Answer: Option (A)

Explanation: Suppose G is an antiderivative for f. The general expression for an antiderivative is $G + C$, where C is constant. We see that for options (b), (c), and (d), it is always possible to solve the equation we obtain to get one or more real values of C. However, (a) simplifies to $G(1) + C = G(0) + C$, whereby C is canceled, and we are left with the statement $G(1) = G(0)$. If this statement is true, then *all* choices of C work, and if it is false, then *none* works. Since we cannot guarantee the truth of the statement, (a) is the exceptional condition.

Another way of thinking about this is that $F(1) - F(0) = \int_0^1 f(x)\,dx$, regardless of the choice of F. If this integral is 0, then any antiderivative works. If it is not zero, no antiderivative works.

Performance review: 17 out of 25 people got this. 5 chose (D), 2 chose (E), 1 chose (B).

Historical note (earlier appeance this quarter): 18 out of 24 got this. 3 chose (C), 2 chose (E), 1 chose (B).

(4) Suppose F is a function defined on $\mathbb{R} \setminus \{0\}$ such that $F'(x) = -1/x^2$ for all $x \in \mathbb{R} \setminus \{0\}$. Which of the following pieces of information is/are **sufficient** to determine F completely? Please see options (D) and (E) before answering.
(A) The value of F at any two positive numbers.
(B) The value of F at any two negative numbers.
(C) The value of F at a positive number and a negative number.
(D) Any of the above pieces of information is sufficient, i.e., we need to know the value of F at any two numbers.
(E) None of the above pieces of information is sufficient.

Answer: Option (C)

Explanation: There are two open intervals: $(-\infty, 0)$ and $(0, \infty)$, on which we can look at F. On each of these intervals, $F(x) = 1/x+$ a constant, but the constant for $(-\infty, 0)$ may differ from the constant for $(0, \infty)$. Thus, we need the initial value information at one positive number and one negative number.

Performance review: 19 out of 25 people got this. 6 chose (D).

Historical note (earlier appearance this quarter): 16 out of 24 got this. 8 chose (D).

(5) Suppose F, G are continuously differentiable functions defined on all of $\mathbb{R}$. Suppose a, b are real numbers with $a < b$. Suppose, further, that $G(x)$ is identically zero everywhere except on the open interval (a, b). Then, what can we say about the relationship between the numbers $P = \int_a^b F(x)G'(x)\,dx$ and $Q = \int_a^b F'(x)G(x)\,dx$?
(A) $P = Q$
(B) $P = -Q$
(C) $PQ = 0$
(D) $P = 1 - Q$
(E) $PQ = 1$

Answer: Option (B)

Explanation: Integration by parts gives us that:

$$\int_a^b F(x)G'(x)\,dx = [F(x)G(x)]_a^b - \int_a^b F'(x)G(x)\,dx$$

Since $G(x) = 0$ outside (a, b), we get that $G(a) = G(b) = 0$, so that the evaluation of $[F(x)G(x)]_a^b$ gives 0. We are thus left with:

$$P = -Q$$

Performance review: 19 out of 25 people got this. 3 chose (D), 2 chose (A), 1 chose (C).

Historical note (Math 153): 32 out of 41 got this. 5 chose (A), 2 chose (C), 1 chose (D), 1 wrote multiple options.

(6) Consider the integration $\int p(x)q''(x)\,dx$. Apply integration by parts twice, first taking p as the part to differentiate, and q as the part to integrate, and then again apply integration by parts to avoid a circular trap. What can we conclude?
(A) $\int p(x)q''(x)\,dx = \int p''(x)q(x)\,dx$

(B) $\int p(x)q''(x)\,dx = \int p'(x)q'(x)\,dx - \int p''(x)q(x)\,dx$
(C) $\int p(x)q''(x)\,dx = p'(x)q'(x) - \int p''(x)q(x)\,dx$
(D) $\int p(x)q''(x)\,dx = p(x)q'(x) - p'(x)q(x) + \int p''(x)q(x)\,dx$
(E) $\int p(x)q''(x)\,dx = p(x)q'(x) - p'(x)q(x) - \int p''(x)q(x)\,dx$

Answer: Option (D)

Explanation: Just write it out.

Performance review: 21 out of 25 people got this. 2 chose (C), 1 each chose (B) and (E).

Historical note (Math 153): 35 out of 41 got this. 4 chose (E) (sign error, didn't notice double negative), 2 chose (A).

(7) Suppose p is a polynomial function. In order to find the indefinite integral for a function of the form $x \mapsto p(x)\exp(x)$, the general strategy, which always works, is to take $p(x)$ as the part to differentiate and $\exp(x)$ as the part to integrate, and keep repeating the process. Which of the following is the best explanation for why this strategy works?

(A) exp can be repeatedly differentiated (staying exp) and polynomials can be repeatedly integrated (giving polynomials all the way).
(B) exp can be repeatedly integrated (staying exp) and polynomials can be repeatedly differentiated, eventually becoming zero.
(C) exp and polynomials can both be repeatedly differentiated.
(D) exp and polynomials can both be repeatedly integrated.
(E) We need to use the recursive version of integration by parts whereby the original integrand reappears after a certain number of applications of integration by parts (i.e., the polynomial equals one of its higher derivatives, up to sign and scaling).

Answer: Option (B)

Explanation: This follows because the polynomial is the part that we are choosing to differentiate.

Performance review: 20 out of 25 people got this. 2 each chose (C) and (E). 1 left the question blank.

Historical note (Math 153): All 41 got this.

(8) Consider the function $x \mapsto \exp(x)\sin x$. This function can be integrated using integration by parts. What can we say about how integration by parts works?

(A) We choose exp as the part to integrate and sin as the part to differentiate, and apply this process once to get the answer directly.
(B) We choose exp as the part to integrate and sin as the part to differentiate, and apply this process once, then use a *recursive* method (identify the integrals on the left and right side) to get the answer.
(C) We choose exp as the part to integrate and sin as the part to differentiate, and apply this process twice to get the answer directly.
(D) We choose exp as the part to integrate and sin as the part to differentiate, and apply this process twice, then use a *recursive* method (identify the integrals on the left and right side) to get the answer.
(E) We choose exp as the part to integrate and sin as the part to differentiate, and we apply integration by parts four times to get the answer directly.

Answer: Option (D)

Explanation: sin is the negative of its second derivative, exp equals its second antiderivative.

Performance review: 22 out of 25 people got this. 3 chose (C).

Historical note (Math 153): 38 out of 41 got this. 3 chose (C).

(9) Suppose f is a continuous function on all of $\mathbb{R}$ and is the third derivative of an elementarily expressible function, but is not the fourth derivative of any elementarily expressible function. In other words, f can be integrated three times but not four times within the collection of elementarily expressible functions. What is the **largest positive integer** k such that $x \mapsto x^k f(x)$ is *guaranteed to be* **elementarily integrable**?

(A) 1
(B) 2
(C) 3

(D) 4
(E) 5

Answer: Option (B)

Explanation: Via integration by parts, integrating f m times is equivalent to finding antiderivatives for $f(x)$, $xf(x)$, and so on till $x^{m-1}f(x)$. In our case, f can be integrated 3 times, so the largest k is $3 - 1 = 2$.

Performance review: 21 out of 25 people got this. 3 chose (C), 1 chose (D).

Historical note (last time): 5 out of 18 people got this correct. 6 chose (D), 5 chose (C), and 2 chose (E).

(10) Suppose f is a continuous function on $(0, \infty)$ and is the third derivative of an elementarily expressible function, but is not the fourth derivative of any elementarily expressible function. In other words, f can be integrated three times but not four times within the collection of elementarily expressible functions. What is the **largest positive integer** k such that the function $x \mapsto f(x^{1/k})$ with domain $(0, \infty)$ is *guaranteed to be* **elementarily integrable**?
(A) 1
(B) 2
(C) 3
(D) 4
(E) 5

Answer: Option (C)

Explanation: Via the u-substitution $u = x^{1/k}$, we get $\int ku^{k-1}f(u)\,du$. Now using the previous question, the maximum value of $k - 1$ possible is 2, so the maximum possible value is 3.

We can also do a direct integration by parts taking 1 as the second part.

Performance review: 21 out of 25 people got this. 2 each chose (A) and (B).

Historical note (last time): 8 out of 18 people got this correct. 5 chose (A), 3 chose (D), and 2 chose (B).

(11) Of these five functions, four of the functions are elementarily integrable and can be integrated using integration by parts. The other one function is **not elementarily integrable**. Identify this function.
(A) $x \mapsto x \sin x$
(B) $x \mapsto x \cos x$
(C) $x \mapsto x \tan x$
(D) $x \mapsto x \sin^2 x$
(E) $x \mapsto x \tan^2 x$

Answer: Option (C)

Explanation: If f is elementarily integrable, then $xf(x)$ is elementarily integrable iff f is twice elementarily integrable; this is easily seen using integration by parts. Of the function options given here, tan is the only function that is not twice elementarily integrable, because the first integration gives $-\ln|\cos x|$ which cannot be integrated. Of the others, note that sin, cos, and $\sin^2$ can be integrated using elementary functions infinitely many times. $\tan^2$ is twice elementarily integrable but no further: integrates the first time to $\tan x - x$, which integrates one more time to $-\ln|cosx| - x^2/2$, which cannot be integrated further.

Performance review: 20 out of 25 people got this. 2 each chose (D) and (E), 1 chose (B).

Historical note (last time): 8 out of 18 people got this correct. 8 people chose (E) and 2 people chose (D).

(12) Consider the four functions $f_1(x) = \sqrt{\sin x}$, $f_2(x) = \sin\sqrt{x}$, $f_3(x) = \sin^2 x$ and $f_4(x) = \sin(x^2)$, all viewed as functions on the interval $[0, 1]$ (so they are all well defined). Two of these functions are elementarily integrable; the other two are not. Which are **the two elementarily integrable functions**?
(A) f_3 and f_4.
(B) f_1 and f_3.
(C) f_1 and f_4.
(D) f_2 and f_3.
(E) f_2 and f_4.

Answer: Option (D)

Explanation: Integration of f_3 is a standard procedure, so we say nothing about that. As for f_2, recall that integrating $f(x^{1/k})$ is equivalent to integrating $u^{k-1}f(u)$ where $u = x^{1/k}$, which in turn is equivalent to integrating f k times. Since sin can be integrated as many times as we wish, f_2 can be integrated.

The reason why f_1 and f_4 are not elementarily integrable is subtler but it's clear that none of the obvious methods work.

Performance review: 20 out of 25 people got this. 2 chose (B), 1 each chose (A), (C), and (E).

Historical note (last time): 5 out of 18 people got this correct. 5 people each chose (B) and (E), 2 chose (C), 1 chose (A).

(13) Which of the following functions has an antiderivative that is **not equivalent** up to elementary functions to the antiderivative of $x \mapsto e^{-x^2}$?

(A) $x \mapsto e^{-x^4}$
(B) $x \mapsto e^{-x^{2/3}}$
(C) $x \mapsto e^{-x^{2/5}}$
(D) $x \mapsto x^2 e^{-x^2}$
(E) $x \mapsto x^4 e^{-x^2}$

Answer: Option (A)

Explanation: We show the equivalence with the others.

Option (D): We use integration by parts, writing $x^2e^{-x^2}$ as $x \cdot (xe^{-x^2})$ and taking xe^{-x^2} as the part to integrate, so that x is the part to differentiate. An antiderivative for xe^{-x^2} is $(-1/2)e^{-x^2}$, so we get:

$$\frac{-x}{2}e^{-x^2} - \int \frac{-1}{2}e^{-x^2}\,dx$$

We thus see that it reduces to $\int e^{-x^2}\,dx$.

Option (E), via reduction to option (D): We use integration by parts, taking x^3 as the part to differentiate and xe^{-x^2} as the part to integrate. One application of integration by parts reduces this to $\int x^2e^{-x^2}$, which is option (D).

Option (B), via reduction to option (D): Start with $\int e^{-x^{2/3}}\,dx$. Put $u = x^{1/3}$. The substitution gives (up to scalars) $\int u^2e^{-u^2}\,du$, which is option (D).

Option (C), via reduction to option (D): Start with $\int e^{-x^{2/5}}\,dx$. Put $u = x^{1/5}$. The substitution gives (up to scalars) $\int u^4e^{-u^2}\,du$, which is option (E).

Performance review: 22 out of 25 people got this. 1 each chose (B), (D), and (E).

Historical note (last time): 6 out of 18 people got this correct. 6 chose (C), 2 each chose (B), (D), (E).

(14) Consider the statements P and Q, where P states that every rational function is elementarily integrable, and Q states that any rational function is k times elementarily integrable for all positive integers k.

Which of the following additional observations is **correct** and **allows us to deduce** Q given P?

(A) There is no way of deducing Q from P because P is true and Q is false.
(B) The antiderivative of a rational function can always be chosen to be a rational function, hence Q follows from a repeated application of P.
(C) Using integration by parts, we see that repeated integration of a function f is equivalent to integrating f, f^2, f^3, and higher powers of f (the powers here are pointwise products, not compositions). If f is a rational function, each of these is also a rational function. Applying P, each of these is elementarily integrable, hence f is k times elementarily integrable for all k.
(D) Using integration by parts, we see that repeated integration of a function f is equivalent to integrating f, f', f'', and higher derivatives of f. If f is a rational function, each of these is also a rational function. Applying P, each of these is elementarily integrable, hence f is k times elementarily integrable for all k.

(E) Using integration by parts, we see that repeated integration of a function f is equivalent to integrating each of the functions $f(x)$, $xf(x)$, $\dots$. If f is a rational function, each of these is also a rational function. Applying P, each of these is elementarily integrable, hence f is k times elementarily integrable for all k.

Answer: Option (E)

Explanation: Review the material on rational function integration.

Performance review: 21 out of 25 got this. 2 chose (D), 1 each chose (B) and (C).

Historical note (last time): 3 out of 18 people got this correct. 7 chose (C), 6 chose (D), 1 each chose (A) and (B).

(15) Which of these functions of x is *not* elementarily integrable?

(A) $x\sqrt{1+x^2}$
(B) $x^2\sqrt{1+x^2}$
(C) $x(1+x^2)^{1/3}$
(D) $x\sqrt{1+x^3}$
(E) $x^2\sqrt{1+x^3}$

Answer: Option (D)

—em Explanation: For options (A) and (C), the substitution $u = 1 + x^2$ works fine. For option (E), the substitution $u = 1 + x^3$ works fine. For option (B), we can solve the problem using a trigonometric substitution. This leaves option (D) (which, incidentally, requires the use of elliptic integrals).

Performance review: 22 out of 25 got this. 3 chose (C).

Historical note (last time): 9 out of 18 people got this correct. 4 chose (C), 4 chose (B), 1 chose (A).

(16) Consider the function $f(k) := \int_1^2 \frac{dx}{\sqrt{x^2+k}}$. f is defined for $k \in (-1, \infty)$. What can we say about the nature of f within this interval?

(A) f is increasing on the interval $(-1, \infty)$.
(B) f is decreasing on the interval $(-1, \infty)$.
(C) f is increasing on $(-1, 0)$ and decreasing on $(0, \infty)$.
(D) f is decreasing on $(-1, 0)$ and increasing on $(0, \infty)$.
(E) f is increasing on $(-1, 0)$, decreasing on $(0, 2)$, and increasing again on $(2, \infty)$.

Answer: Option (B)

Explanation: For any fixed value of $x \in [1, 2]$, the integrand $1/\sqrt{x^2 + k}$ is a *decreasing* function of k for $k \in (-1, \infty)$. Hence, the value we get upon integrating it for $x \in [1, 2]$ should also be a decreasing function of k.

Performance review: 15 out of 25 got this. 6 chose (D), 3 chose (C), 1 chose (A).

Historical note (last time): 6 out of 18 people got this correct. 3 chose (A), 5 chose (C), 2 chose (E), 1 chose (D), 1 left the question blank.

(17) For which of these functions of x does the antiderivative necessarily involve *both* arctan *and* ln?

(A) $1/(x + 1)$
(B) $1/(x^2 + 1)$
(C) $x/(x^2 + 1)$
(D) $x/(x^3 + 1)$
(E) $x^2/(x^3 + 1)$

Answer: Option (D)

Explanation: Option (A) integrates to $\ln|x + 1|$, option (B) integrates to $\arctan x$, option (C) integrates to $(1/2)\ln(x^2 + 1)$, and option (E) integrates to $(1/3)\ln|x^3 + 1|$. For option (D), we need to use partial fractions with denominators $x + 1$ and $x^2 - x + 1$, and we end up getting nonzero coefficients on terms that integrate to ln and to arctan.

Performance review: 21 out of 25 people got this. 2 chose (E), 1 each chose (B) and (C).

Historical note (last time): 13 out of 18 people got this correct. 3 chose (C), 1 each chose (B) and (E).

(18) Suppose F is a (not known) function defined on $\mathbb{R} \setminus \{-1, 0, 1\}$, differentiable everywhere on its domain, such that $F'(x) = 1/(x^3 - x)$ everywhere on $\mathbb{R} \setminus \{-1, 0, 1\}$. For which of the following sets of points is it true that knowing the value of F at these points **uniquely** determines F?
(A) $\{-\pi, -e, 1/e, 1/\pi\}$
(B) $\{-\pi/2, -\sqrt{3}/2, 11/17, \pi^2/6\}$
(C) $\{-\pi^3/7, -\pi^2/6, \sqrt{13}, 11/2\}$
(D) Knowing F at any of the above determines the value of F uniquely.
(E) None of the above works to uniquely determine the value of F.

Answer: Option (B)

Explanation: The domain of F has four connected components: the open intervals $(-\infty, -1)$, $(-1, 0)$, $(0, 1)$, and $(1, \infty)$. We need to know the value of F at one point in each of these intervals. By computing values, we see that the set of points in option (B) has the property that it contains one point in each of these intervals, and those in options (A) and (C) do not.

Performance review: 17 out of 25 people got this. 4 chose (A), 2 chose (E), 1 each chose (C) and (D).

Historical note (last time): 8 out of 18 people got this correct. 5 chose (D), 3 chose (A), 2 left the question blank.

(19) Suppose F is a continuously differentiable function whose domain contains (a, ∞) for some $a \in \mathbb{R}$, and $F'(x)$ is a rational function $p(x)/q(x)$ on the domain of F. Further, suppose that p and q are nonzero polynomials. Denote by d_p the degree of p and by d_q the degree of q. Which of the following is a **necessary and sufficient condition** to ensure that $\lim_{x \to \infty} F(x)$ is finite?
(A) $d_p - d_q \geq 2$
(B) $d_p - d_q \geq 1$
(C) $d_p = d_q$
(D) $d_q - d_p \geq 1$
(E) $d_q - d_p \geq 2$

Answer: Option (E)

Explanation: This can be justified in terms of partial fractions. Th ecase where q is a product of linear factors can be justified using the previous question. But that is not the most elegant justification. When we cover sequences and series, we will see some comparison tests that make it clear why this holds. The basic example you can keep in mind is that the antiderivative of $1/x^2$ is $-1/x$, which has a finite limit as $x \to \infty$.

Performance review: 9 out of 25 people got tihs. 13 chose (D), 2 chose (C), 1 chose (B).

Historical note (last time): 3 out of 18 people got this correct. 7 chose (D), 5 chose (B), 2 chose (C), 1 chose (A).

Those who chose (D) had the right idea but failed to account for the extra margin that needs to be maintained because an integration is being performed.

For the next two questions, build on the observation: For any nonconstant monic polynomial $q(x)$, there exists a finite collection of transcendental functions $f_1, f_2, \dots, f_r$ such that the antiderivative of any rational function $p(x)/q(x)$, on an open interval where it is defined and continuous, can be expressed as $g_0 + f_1 g_1 + f_2 g_2 + \cdots + f_r g_r$ where $g_0, g_1, \dots, g_r$ are rational functions.

(20) For the polynomial $q(x) = 1 + x^2$, what collection of f_is works (all are written as functions of x)?
(A) $\arctan x$ and $\ln |x|$
(B) $\arctan x$ and $\arctan(1 + x^2)$
(C) $\ln |x|$ and $\ln(1 + x^2)$
(D) $\arctan x$ and $\ln(1 + x^2)$
(E) $\ln |x|$ and $\arctan(1 + x^2)$

Answer: Option (D)

Explanation: Follows from the standard partial fraction decomposition. $2x/(1 + x^2)$ gives the ln integration and $1/(1 + x^2)$ gives the arctan integration.

Performance review: 23 out of 25 people got this. 1 each chose (A) and (C).

Historical note (last time): 5 out of 18 people got this correct. 4 each chose (C) and (E), 3 chose (A), 2 chose (B).

(21) For the polynomial $q(x) := 1 + x^2 + x^4$, what is the size of the smallest collection of f_is that works?
(A) 1
(B) 2
(C) 3
(D) 4
(E) 5

Answer: Option (D)

Explanation: The denominator factors into $x^2 - x + 1$ and $x^2 + x + 1$. Each of these contributes one arctan possibility and one ln possibility. A total of 4 possibilities is achieved.

In general, if there are no repeated factors, the smallest number of pieces equals the degree of the polynomial.

Performance review: 17 out of 25 people got this. 8 chose (B).

Historical note (Math 153): 30 out of 44 got this. 8 chose (B), 5 chose (C), 1 chose (E).

TAKE-HOME CLASS QUIZ SOLUTIONS: DUE FRIDAY FEBRUARY 22: MULTI-VARIABLE INTEGRATION

MATH 195, SECTION 59 (VIPUL NAIK)

1. Performance review

24 people took this quiz. The score distribution was as follows:

- Score of 2: 1 person
- Score of 3: 5 people
- Score of 4: 7 people
- Score of 5: 7 people
- Score of 6: 2 people
- Score of 7: 2 people

The question wise answers and performance review are as follows:

(1) Option (C): 14 people
(2) Option (A): 19 people
(3) Option (C): 17 people
(4) Option (D): 14 people
(5) Option (A): 7 people
(6) Option (C): 14 people
(7) Option (C): 20 people
(8) Option (D): 1 person

2. Solutions

The following setup is for the first five questions only.

Suppose F is a function of two real variables, say x and t, so $F(x,t)$ is a real number for x and t restricted to suitable open intervals in the real number. Suppose, further, that F is jointly continuous (whatever that means) in x and t.

Define $f(t) := \int_0^\infty F(x,t)\,dx$. Here, while doing the integration, t is treated as a constant. x, the variable of integration, is being integrated on $[0,\infty)$.

Suppose further that f is defined and continuous for t in $(0,\infty)$.

In the next few questions, you are asked to compute the function f explicitly given the function F, for $t \in (0,\infty)$.

(1) *Do not discuss!* $F(x,t) := e^{-tx}$. Find f. *Last time:* 15/19 *correct*
(A) $f(t) = e^{-t}/t$
(B) $f(t) = e^{t}/t$
(C) $f(t) = 1/t$
(D) $f(t) = -1/t$
(E) $f(t) = -t$

Answer: Option (C)

Explanation: The integral becomes $[-e^{-tx}/t]_0^\infty$. Plugging in at ∞ gives 0 and plugging in at 0 gives $-1/t$. Since the value at 0 is being subtracted, we eventually get $1/t$.

Note that the answer must be positive for the simple reason that we are integrating a positive function from left to right across an interval.

Performance review: 14 out of 24 people got this. 9 chose (D), 1 chose (E).

Historical note (last time): 15 out of 19 people got this correct. 3 people chose (D) and 1 person chose (B).

(2) *Do not discuss!* $F(x,t) := 1/(t^2+x^2)$. Find f. *Last time:* 13/19 *correct*
(A) $f(t) = \pi/(2t)$
(B) $f(t) = \pi/t$
(C) $f(t) = 2\pi/t$
(D) $f(t) = \pi t$
(E) $f(t) = 2\pi t$
Answer: Option (A)
Explanation: We get $[(1/t)\arctan(x/t)]_0^\infty$. The evaluation at ∞ gives $\pi/(2t)$ and the evaluation at 0 gives 0. Subtracting, we get $\pi/(2t)$.
Performance review: 19 out of 24 people got this. 3 chose (B) and 2 chose (C).
Historical note (last time): 13 out of 19 people got this correct. 2 people each chose (B), (C), and (D).

(3) *Do not discuss!* $F(x,t) := 1/(t^2+x^2)^2$. Find f. *Last time:* 13/19 *correct*
(A) $f(t) = \pi/t^3$
(B) $f(t) = \pi/(2t^3)$
(C) $f(t) = \pi/(4t^3)$
(D) $f(t) = \pi/(8t^3)$
(E) $f(t) = 3\pi/(8t^3)$
Answer: Option (C)
Explanation: Put in $\theta = \arctan(x/t)$. Substitute, and we get $(1/t^3)\int_0^{\pi/2}\cos^2\theta\,d\theta$. Integrating, we get $[\theta/2t^3 + \sin(2\theta)/4t^3]_0^{\pi/2}$. The trigonometric part vanishes between limits, and we are left with $\pi/(4t^3)$
Performance review: 17 out of 24 people got this. 6 chose (B), 1 chose (D).
Historical note (last time): 13 out of 19 people got this correct. 3 people chose (D) and 1 each chose (A), (B), and (E).

(4) (*) *You can discuss this!* $F(x,t) = \exp(-(tx)^2)$. Use that $\int_0^\infty \exp(-x^2)\,dx = \sqrt{\pi}/2$. Find f. *Last time:* 7/19 *correct*
(A) $f(t) = t^2\sqrt{\pi}/2$
(B) $f(t) = t\sqrt{\pi}/2$
(C) $f(t) = \sqrt{\pi}/2$
(D) $f(t) = \sqrt{\pi}/(2t)$
(E) $f(t) = \sqrt{\pi}/(2t^2)$
Answer: Option (D)
Explanation: Put $u = tx$, get a $1/t$ on the outside, giving $(1/t)\int_0^\infty \exp(-u^2)\,du$.
Performance review: 14 out of 24 got this, 7 chose (B), 2 chose (E), 1 chose (C).
Historical note (last time): 7 out of 19 people got this correct. 7 people chose (E), 2 each chose (A) and (B), and 1 chose (C).

(5) (**) *You can discuss this!* (could confuse you if you don't understand it): In the same general setup as above (but with none of these specific Fs), which of the following is a *sufficient* condition for f to be an increasing function of t? *Last time:* 3/19 *correct*
(A) $t \mapsto F(x_0,t)$ is an increasing function of t for every choice of $x_0 \geq 0$.
(B) $x \mapsto F(x,t_0)$ is an increasing function of x for every choice of $t_0 \in (0,\infty)$.
(C) $t \mapsto F(x_0,t)$ is a decreasing function of t for every choice of $x_0 \geq 0$.
(D) $x \mapsto F(x,t_0)$ is a decreasing function of x for every choice of $t_0 \in (0,\infty)$.
(E) None of the above.
Answer: Option (A)
Explanation: If F is increasing in t for every value of x_0, then that means that as t gets bigger, the function F being integrated gets bigger everywhere in x, i.e., if $t_1 < t_2$, then $F(t_1,x_0) < F(t_2,x_0)$ for every $x_0 \geq 0$. The integral for the larger value t_2 must therefore also be bigger. (We looked at this stuff in Section 5.8 of the book).

Performance review: 7 out of 24 got this. 9 chose (C), 5 chose (B), 2 chose (D), 1 chose (E).

Historical note (last time): 3 out of 19 people got this correct. 9 chose (B), 4 chose (E), 2 chose (C), 1 chose (D).

(end of the setup)

(6) *Do not discuss!* Suppose f is a homogeneous polynomial of degree $d > 0$. Define g as the following function on positive reals: $g(a)$ is the double integral of f on the square $[0, a] \times [0, a]$. Assuming that $g(a)$ is not identically the zero function, which of these best describes the nature of $g(a)$? *Last time: 12/19 correct.*
(A) A constant times a^d
(B) A constant times a^{d+1}
(C) A constant times a^{d+2}
(D) A constant times a^{2d+1}
(E) A constant times a^{2d+2}

Answer: Option (C)

Explanation: Each monomial is of the form a constant times $x^p y^q$ where $p + q = d$. Integrating this as a multiplicatively separable function gives $x^{p+1}y^{q+1}$ times a constant. Evaluating between limits gives a^{d+2} times a constant. This is the form of the double integral of each monomial, and hence the double integral of the sum is also of the same form.

Performance review: 14 out of 24 people got tihs. 5 each chose (B) and (E).

Historical note (last time): 12 out of 19 people got this correct. 3 people chose (D), 2 each chose (A) and (E).

(7) (*) *You can discuss this!* Suppose $g(x, y)$ and $G(x, y)$ are continuous functions of two variables and $G_{xy} = g$. How can the double integral $\int_s^t \int_u^v g(x, y)\, dy\, dx$ be described in terms of the values of G? *Last time: 8/19 correct*
(A) $G(v, t) + G(u, s) - G(u, t) - G(v, s)$
(B) $G(v, t) - G(v, s) + G(u, t) - G(u, s)$
(C) $G(t, v) + G(s, u) - G(t, u) - G(s, v)$
(D) $G(t, v) - G(s, v) + G(t, u) - G(s, u)$
(E) $G(t, v) + G(v, t) - G(s, u) - G(u, s)$

Answer: Option (C)

Explanation: Note that the integration is over $[s, t] \times [u, v]$, i.e., the rectangular region with corner points (s, u), (s, v), (t, u), and (t, v). Recall that for the double integral, we put positive signs on the two extreme points (top right and bottom left) and negative signs on the other two points (top left and bottom right). See more in the lecture notes.

Performance review: 20 out of 24 people got this. 2 chose (D), 1 each chose (A) and (B).

Historical note (last time): 8 out of 19 people got this correct. 6 people chose (D), 2 each chose (A) and (E), 1 chose (B).

(8) (**) *You can discuss this!* Suppose f is an elementarily integrable function, but $f(x^k)$ is not elementarily integrable for any integer $k > 1$ (examples are sin, exp, cos). For which of the following types of regions D are we *guaranteed to be able* to compute, in elementary function terms, the double integral $\int_D \int f(x^2)\, dA$ over the region (note that f is just a function of x, but we treat it as a function of two variables)? Please see Option (E) before answering and select that if applicable. *Last time: 1/19 correct.*
(A) A rectangle with vertices $(0, 0)$, $(0, b)$, $(a, 0)$, and (a, b), with $a, b > 0$.
(B) A triangle with vertices $(0, 0)$, $(0, b)$, $(a, 0)$, with $a, b > 0$.
(C) A triangle with vertices $(0, 0)$, $(0, b)$, (a, b), with $a, b > 0$.
(D) A triangle with vertices $(0, 0)$, $(a, 0)$, (a, b), with $a, b > 0$.
(E) All of the above

Answer: Option (D)

Explanation: For such a triangle, we integrate on y inner and x outer. For a fixed value of x, the y-value ranges from 0 to bx/a, so the integral becomes:

$$\int_0^a \int_0^{bx/a} f(x^2)\, dy\, dx$$

$f(x^2)$ pulls out of the inner integral and the inner integral just gives (bx/a), so we get:

$$\int_0^a \frac{b}{a} x f(x^2)\, dx$$

This can be done by the substitution $u = x^2$ and the knowledge that f is elementarily integrable. All the other integrations stumble because they require knowledge of an antiderivative of $f(x^2)$.

Performance review: 1 out of 24 people got this. 7 chose (A), 6 each chose (B) and (E), and 4 chose (C).

Historical note (last time): 1 person got this correct. 8 chose (E), 5 chose (A), 3 chose (B), 2 chose (C).

TAKE-HOME CLASS QUIZ SOLUTIONS: DUE WEDNESDAY FEBRUARY 27: TAYLOR SERIES AND POWER SERIES

MATH 195, SECTION 59 (VIPUL NAIK)

1. Performance review

24 people took this 17-question quiz. The score distribution was as follows:

- Score of 3: 1 person
- Score of 5: 1 person
- Score of 6: 2 people
- Score of 8: 1 person
- Score of 13: 1 person
- Score of 15: 1 person
- Score of 16: 5 people
- Score of 17: 12 people

(1) Option (C): 18 people
(2) Option (C): 21 people
(3) Option (C): 20 people
(4) Option (D): 22 people
(5) Option (E): 22 people
(6) Option (E): 18 people
(7) Option (C): 17 people
(8) Option (B): 18 people
(9) Option (D): 20 people
(10) Option (E): 19 people
(11) Option (C): 20 people
(12) Option (A): 22 people
(13) Option (D): 19 people
(14) Option (D): 22 people
(15) Option (A): 21 people
(16) Option (D): 19 people
(17) Option (D): 22 people

2. Solutions

For these questions, we denote by $C^\infty(\mathbb{R})$ the space of functions from $\mathbb{R}$ to $\mathbb{R}$ that are *infinitely* differentiable *everywhere* in $\mathbb{R}$.

We denote by $C^k(\mathbb{R})$ the space of functions from $\mathbb{R}$ to $\mathbb{R}$ that are at least k times continuously differentiable on all of $\mathbb{R}$. Note that for $k \geq l$, $C^k(\mathbb{R})$ is a subspace of $C^l(\mathbb{R})$. Further, $C^\infty(\mathbb{R})$ is the intersection of $C^k(\mathbb{R})$ for all k.

We say that a function f is analytic about c if the Taylor series of f about c converges to f on some open interval about c. We say that f is *globally analytic* if the Taylor series of f about 0 converges to f everywhere on $\mathbb{R}$.

It turns out that if a function is globally analytic, then it is analytic not only about 0 but about any other point. In particular, globally analytic functions are in $C^\infty(\mathbb{R})$.

(1) Recall that if f is a function defined and continuous around c with the property that $f(c) = 0$, the order of the zero of f at c is defined as the least upper bound of the set of real β for which

$\lim_{x\to c} |f(x)|/|x-c|^\beta = 0$. If f is in $C^\infty(\mathbb{R})$, what can we conclude about the orders of zeros of f? *Two years ago:* 11/26 *correct*

(A) The order of any zero of f must be between 0 and 1.
(B) The order of any zero of f must be between 1 and 2.
(C) The order of any zero of f, if finite, must be a positive integer.
(D) The order of any zero of f must be exactly 1.
(E) The order of any zero of f must be ∞.

Answer: Option (C)

Explanation: We consider two cases. First, that for every positive integer k, we have $f^{(k)}(c) = 0$. In that case, we can verify using the LH rule that $f(x)/(x-c)^k \to 0$ for every positive integer k, and hence, there is no finite least upper bound and hence no finite order.

Next, suppose there is a smallest k such that $f^{(k)}(c) \neq 0$. Suppose $f^{(k)}(c) = \lambda$. This k must be greater than 0, because we are given that $f^{(0)}(c) = f(c) = 0$. We can show by a k-fold application of the LH rule that $\lim_{x\to c} f(x)/(x-c)^k = \lambda/k!$ which is a finite nonzero number. By suitable chaining, we can therefore show that $\lim_{x\to c} |f(x)|/|x-c|^\beta = 0$ for all $\beta \in (0,k)$. Thus, the order of the zero at f is precisely k, whuch is a positive integer.

Performance review: 18 out of 24 people got this. 3 chose (E), 2 chose (A), 1 chose (D).

Historical note (Math 153): 32 out of 40 got this. 7 chose (E), 1 chose (B).

Historical note (last year): 3 out of 11 people got this. 3 chose (B), 4 chose (E), and 1 left the question blank.

Historical note (two years ago): 11 out of 26 people got this correct. 7 chose (A), 3 each chose (B) and (E), 1 chose (D), and 1 left the question blank.

(2) For the function $f(x) := x^2 + x^{4/3} + x + 1$ defined on $\mathbb{R}$, what can we say about the Taylor polynomials about 0? *Two years ago:* 8/26 *correct*

(A) No Taylor polynomial is defined for f.
(B) $P_0(f)(x) = 1$, $P_n(f)$ is not defined for $n > 0$.
(C) $P_0(f)(x) = 1$, $P_1(f)(x) = 1 + x$, $P_n(f)$ is not defined for $n > 1$.
(D) $P_0(f)(x) = 1$, $P_1(f)(x) = 1 + x$, $P_2(f) = f$, and $P_n(f)$ is not defined for $n > 2$.
(E) $P_0(f)(x) = 1$, $P_1(f)(x) = 1 + x$, $P_2(f) = f$, and $P_n(f) = f$ for all $n > 2$.

Answer: Option (C)

Explanation: The fraction power $x^{4/3}$ can be differentiated once but not twice about 0. The rest of the expression for f is polynomial. Thus, f is once differentiable but not twice differentiable at 0. Hence, we cannot define $P_2(f)$. $P_0(f)$ is just $f(0)$, which is 1, and $P_1(f)(x) = f(0) + f'(0)x = 1 + x$. Alternatively, $P_1(f)(x)$ is simply the truncation of f to the parts of degree at most 1.

Performance review: 21 out of 24 people got this. 2 chose (E), 1 chose (D).

Historical note (Math 153): 34 out of 40 got this. 3 chose (D), 2 chose (E), and 1 chose (B).

Historical note (last year): 2 out of 11 got this correct. 7 chose (E), 1 each chose (B) and (D).

Historical note (two years ago): 8 out of 26 people got this correct. 5 each chose (A) and (B), 4 each chose (D) and (E).

(3) Consider the function $F(x,p) = \sum_{n=1}^\infty x^n/n^p$. For fixed p, this is a power series in x. What can we say about the interval of convergence of this power series about $x = 0$, in terms of p for $p \in (0,\infty)$? *Two years ago:* 4/26 *correct*

(A) The interval of convergence is $(-1,1)$ for $0 < p \le 1$ and $[-1,1]$ for $p > 1$.
(B) The interval of convergence is $(-1,1)$ for $0 < p < 1$ and $[-1,1]$ for $p \ge 1$.
(C) The interval of convergence is $[-1,1)$ for $0 < p \le 1$ and $[-1,1]$ for $p > 1$.
(D) The interval of convergence is $(-1,1]$ for $0 < p < 1$ and $[-1,1]$ for $p \ge 1$.
(E) The interval of convergence is $(-1,1)$ for $0 < p \le 1$ and $[-1,1)$ for $p > 1$.

Answer: Option (C)

Explanation: The radius of convergence is 1 for obvious reasons. Convergence at the boundary point -1 follows from the alternating series theorem. At the boundary point 1, we get a p-series, which converges if and only if $p > 1$.

Performance review: 20 out of 24 got this. 3 chose (A), 1 chose (B).

Historical note (Math 153): 28 out of 40 got this. 5 each chose (A) and (D), 1 chose (B), and 1 left the question blank.

Historical note (last year): 5 out of 11 got this correct. 3 chose (A), 2 chose (D), 1 chose (B).

Historical note (two years ago): 4 out of 26 people got this correct. 10 chose (B), 7 chose (A), 3 chose (E), 2 chose (D).

(4) Which of the following functions of x has a power series $\sum_{k=0}^{\infty} x^{4k}/(4k)!$? *Two years ago: 9/26 correct*
(A) $(\sin x + \sinh x)/2$
(B) $(\sin x - \sinh x)/2$
(C) $(\sinh x - \sin x)/2$
(D) $(\cosh x + \cos x)/2$
(E) $(\cosh x - \cos x)/2$

Answer: Option (D)

Explanation: Take the Taylor series and add. Also, use that both cosh and cos are globally analytic.

Performance review: 22 out of 24 got this. 1 each chose (B) and (C).

Historical note (Math 153): 35 out of 40 got this. 3 chose (A), 1 each chose (B) and (C).

Historical note (last year): 5 out of 11 got this correct. 3 chose (C), 2 chose (A), 1 chose (B).

Historical note (two years ago): 9 out of 26 people got this correct. 5 chose (B), 4 each chose (A), (C), and (E).

(5) What is the sum $\sum_{k=0}^{\infty} (-1)^k x^{2k}/k!$? Note that the denominator is $k!$ and *not* $(2k)!$. *Two years ago: 12/26 correct*
(A) $\cos x$
(B) $\sin x$
(C) $\cos(x^2)$
(D) $\cosh(x^2)$
(E) $\exp(-x^2)$

Answer: Option (E)

Explanation: Put $u = -x^2$, and we get $\sum_{k=0}^{\infty} u^k/k!$.

Performance review: 22 out of 24 got this. 1 each chose (A) and (B).

Historical note (Math 153): 35 out of 40 got this. 5 chose (C).

Historical note (last year): 1 out of 11 got this correct. 7 chose (C), 2 chose (D), 1 chose (B).

Historical note (two years ago): 12 out of 26 people got this correct. 7 chose (C), 3 each chose (A) and (D), 1 chose (B).

(6) Define an operator R from the set of power series about 0 to the set $[0, \infty]$ (nonnegative real numbers along with $+\infty$) that sends a power series $a = \sum a_k x^k$ to the radius of convergence of the power series about 0. For two power series a and b, $a + b$ is the sum of the power series. What can we say about $R(a + b)$ given $R(a)$ and $R(b)$?
(A) $R(a + b) = \max\{R(a), R(b)\}$ in all cases.
(B) $R(a + b) = \min\{R(a), R(b)\}$ in all cases.
(C) $R(a+b) = \max\{R(a), R(b)\}$ if $R(a) \neq R(b)$. If $R(a) = R(b)$, then $R(a+b)$ could be any number greater than or equal to $\max\{R(a), R(b)\}$.
(D) $R(a+b) = \max\{R(a), R(b)\}$ if $R(a) \neq R(b)$. If $R(a) = R(b)$, then $R(a+b)$ could be any number less than or equal to $\max\{R(a), R(b)\}$.
(E) $R(a+b) = \min\{R(a), R(b)\}$ if $R(a) \neq R(b)$. If $R(a) = R(b)$, then $R(a+b)$ could be any number greater than or equal to $\min\{R(a), R(b)\}$.

Answer: Option (E)

Explanation: $R(a + b) \geq \min\{R(a), R(b)\}$ because if both a and b converge, so does $a + b$. Let $c = a + b$. We also then get that $R(a) \geq \min\{R(b), R(c)\}$ and $R(b) \geq \min\{R(a), R(c)\}$ because $a = c - b$ and $b = c - a$.

Juggling these possibilities, we find that of the three numbers $R(a)$, $R(b)$, and $R(a+b)$, the smaller two of the three numbers must be equal. This forces option (E).

This is a type of hyperbolic geometry – all "triangles" must be isosceles.

Performance review: 18 out of 24 got this. 3 chose (C), 2 chose (A), 1 chose (B).

Historical note (Math 153): 29 out of 40 got this. 5 chose (D), 3 each chose (B) and (C).

Historical note (last year): 3 out of 11 got this correct. 4 chose (B) 3 chose (D), 1 chose (C).

Historical note (two years ago): 3 out of 26 people got this correct. 11 chose (C), 7 chose (D), 3 chose (A), 1 chose (B).

(7) Which of the following is/are true? *Two years ago:* 5/26 *correct*
- (A) If we start with any function in $C^\infty(\mathbb{R})$ and take the Taylor series about 0, the Taylor series converges everywhere on $\mathbb{R}$.
- (B) If we start with any function in $C^\infty(\mathbb{R})$ and take the Taylor series about 0, the Taylor series converges to the original function on its interval of convergence (which may not be all of $\mathbb{R}$).
- (C) If we start with a power series about 0 that converges everywhere in $\mathbb{R}$, then the function it converges to is in $C^\infty(\mathbb{R})$ and its Taylor series about 0 equals the original power series.
- (D) All of the above.
- (E) None of the above.

Answer: Option (C)

Explanation: See the lecture notes. A counterexample to (A) is arctan, and a counterexample to (B) is e^{-1/x^2}.

Performance review: 17 out of 24 got this. 4 chose (D), 2 chose (B), 1 chose (A).

Historical note (Math 153): 29 out of 40 got this. 6 chose (D), 4 chose (B), 1 chose (A).

Historical note (last year): 9 out of 11 got this correct. 1 chose (B) and 1 chose (E).

Historical note (two years ago): 5 out of 26 people got this correct. 10 chose (B), 5 each chose (D) and (E), and 1 chose (A).

(8) Consider the function $f(x) := \sum_{k=0}^\infty x^k/2^{k^2}$. The power series converges everywhere, so f is a globally analytic function. What is the best description of the manner in which f grows as $x \to \infty$? *Two years ago:* 12/26 *correct*
- (A) f grows polynomially in x.
- (B) f grows faster than any polynomial function but slower than any exponential function of x (i.e., any function of the form $x \mapsto e^{mx}, m > 0$).
- (C) f grows like an exponential function of x, i.e., it can be sandwiched between two exponentially growing functions of x.
- (D) f grows faster than any exponential function but slower than any doubly exponential function of x. Here, doubly exponential means something of the form $e^{ae^{bx}}$ where a and b are both positive.
- (E) f grows like a doubly exponential function of x. Here, doubly exponential means something of the form $e^{ae^{bx}}$ where a and b are both positive.

Answer: Option (B)

Explanation: Note that *any* power series with infinitely many positive coefficients (and no negative coefficients) must grow faster than a polynomial, which, after all, has finite degree. Note that this logic does not work for power series that have a mix of positive and negative coefficients, such as the power series for the sin and cos functions.

The rough reason that growth is strictly slower than an exponential function is that the denominators are growing much faster than $k!$. Recall that if the denominators are $k!$, we get precisely the exponential function. This can be made more precise.

Performance review: 18 out of 24 got this. 4 chose (C), 1 each chose (A) and (E).

Historical note (Math 153): 30 out of 40 got this. 5 chose (A), 3 chose (C), 1 each chose (D) and (E).

Historical note (last year): 5 out of 11 got this correct. 4 chose (D), 1 each chose (A) and (C).

Historical note (two years ago): 12 out of 26 people got this correct. 6 chose (D), 4 chose (C), 2 each chose (A) and (E).

(9) Consider the function $f(x) := \sum_{k=0}^\infty x^k/(k!)^2$. The power series converges everywhere, so the function is globally analytic. What pair of functions bounds f from above and below for $x > 0$? *Two years ago:* 12/26 *correct*

(A) $\exp(x)$ from below and $\cosh(2x)$ from above.
(B) $\exp(x)$ from below and $\cosh(x^2)$ from above.
(C) $\exp(x/2)$ from below and $\exp(x)$ from above.
(D) $\cosh(\sqrt{x})$ from below and $\exp(x)$ from above.
(E) $\cosh(2x)$ from below and $\cosh(x^2)$ from above.

Answer: Option (D)

Explanation: We use the fact that:

$$k! \le (k!)^2 \le (2k)!$$

for all $k \ge 0$, with both inequalities strict if $k \ge 2$.
We thus get:

$$\frac{x^k}{k!} \ge \frac{x^k}{(k!)^2} \ge \frac{x^k}{(2k)!}$$

for $x > 0$, with both inequalities strict if $k \ge 2$. Summing up, we get:

$$\sum_{k=0}^{\infty} \frac{x^k}{k!} > \sum_{k=0}^{\infty} \frac{x^k}{(k!)^2} > \sum_{k=0}^{\infty} \frac{x^k}{(2k)!}$$

The left most expression is e^x. For the right most expression, put $u = \sqrt{x}$, and we get $\cosh u$, so $\cosh\sqrt{x}$. Thus option (D) is the right choice.

Performance review: 20 out of 24 got this. 2 chose (C), 1 each chose (A) and (B).

Historical note (Math 153): 28 out of 40 got this. 7 chose (C), 4 chose (B), 1 chose (A).

Historical note (last year): 1 out of 11 got this correct. 6 chose (C) and 4 chose (B).

Historical note (two years ago): 12 out of 26 people got this correct. 5 each chose (A) and (C), 3 chose (B), and 1 chose (E).

(10) Consider the function $f(x) := \max\{0, x\}$. What can we say about the Taylor series of f centered at various points?
(A) The Taylor series of f centered at any point is the zero series.
(B) The Taylor series of f centered at any point simplifies to x.
(C) The Taylor series of f centered at any point other than zero converges to f globally. However, the Taylor series centered at 0 is not defined.
(D) The Taylor series of f centered at any point is either the zero series or simplifies to x.
(E) The Taylor series of f centered at any point other than the point 0 is either the zero series or simplifies to x. However, the Taylor series is not defined at 0.

Answer: Option (E)

Explanation: A piecewise description of f is:

$$f(x) := \begin{cases} x, & x > 0 \\ 0, & x \le 0 \end{cases}$$

The Taylor series is not defined at 0. The reason is that the function is not differentiable at 0, because the left hand derivative $f'_-(0)$ is 0 and the right hand derivative $f'_+(0)$ is 1.

At any other point, the Taylor series corresponds globally to the piece function for that point. So, the Taylor series at any positive real number is just the x function and the Taylor series at any negative real number is just the 0 function. This ties in with the idea that the Taylor series is completely determined by the local behavior of the function and cannot see changes in the function definition far from the point.

Performance review: 19 out of 24 got this. 2 each chose (B) and (C), 1 chose (A).

Historical note (Math 153): 26 out of 40 got this. 6 chose (C), 4 chose (D), 3 chose (A), 1 chose (B).

(11) Which of the following functions is in $C^\infty(\mathbb{R})$ but is *not* analytic about 0? *Two years ago: 3/26 correct*
(A) $f_1(x) := \begin{cases} (\sin x)/x, & x \neq 0 \\ 1, & x = 0 \end{cases}$

(B) $f_2(x) := \begin{cases} e^{-1/x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$

(C) $f_3(x) := \begin{cases} e^{-1/x^2}, & x \neq 0 \\ 0, & x = 0 \end{cases}$

(D) $f_4(x) := \begin{cases} \sin(1/x), & x \neq 0 \\ 0, & x = 0 \end{cases}$

(E) All of the above.

Answer: Option (C)

Explanation: This answer is explained more in the lecture notes.

Why the other options are wrong:

Option (A): This is in fact globlally analytic, and is given by the power series $1-x^2/3!+x^4/5!-\dots$

Option (B): This is not continuous at 0. The left hand limit at 0 is $+\infty$.

Option (D): This is not continuous at 0. The limit at 0 is not defined.

Performance review: 20 out of 24 got this. 2 each chose (A) and (B).

Historical note (Math 153): 30 out of 42 got this. 5 chose (D), 4 chose (E), 2 chose (B), 1 chose (A).

Historical note (last year): 1 out of 11 got this correct. 5 chose (D), 2 chose (B), 1 each chose (A) and (E).

Historical note (two years ago): 3 out of 26 people got this correct. 11 chose (D), 6 each chose (A) and (E).

(12) Which of the following functions is in $C^\infty(\mathbb{R})$ and is analytic about 0 but is not globally analytic? *Two years ago: 7/26 correct*

(A) $x \mapsto \ln(1+x^2)$

(B) $x \mapsto \ln(1+x)$

(C) $x \mapsto \ln(1-x)$

(D) $x \mapsto \exp(1+x)$

(E) $x \mapsto \exp(1-x)$

Answer: Option (A)

Explanation: The function is in $C^\infty(\mathbb{R})$ because it can be differentiated infinitely often: the first derivative is $2x/(1+x^2)$, and each subsequent derivative is a rational function whose denominator is a power of $1+x^2$. Since $1+x^2$ does not vanish anywhere on $\mathbb{R}$, each derivative is defined and continuous on all of $\mathbb{R}$.

The radius of convergence of the power series is 1, basically because it is a power series where the coefficients are rational functions, and any such power series has radius of convergence 1 by the root test or ratio test. Thus, the function is not globally analytic.

Why the other options are wrong:

Option (B) is not in $C^\infty(\mathbb{R})$ because the function is not defined for $x \leq -1$.

Option (C) is not in $C^\infty(\mathbb{R})$ because the function is not defined for $x \geq 1$.

Options (D) and (E) are globally analytic because exp is globally analytic.

Performance review: 22 out of 24 got this. 1 each chose (C) and (E).

Historical note (Math 153): 28 out of 42 got this. 8 chose (B), 6 chose (C).

Historical note (last year): Nobody got this correct! 8 chose (B), 3 chose (C).

Historical note (two years ago): 7 out of 26 people got this correct. 6 chose (C), 5 chose (B), 4 chose (E), 3 chose (D), and 1 left the question blank.

(13) Suppose f and g are globally analytic functions and g is nowhere zero. Which of the following is *not necessarily* globally analytic?

(A) $f+g$, i.e., the function $x \mapsto f(x)+g(x)$

(B) $f-g$, i.e., the function $x \mapsto f(x)-g(x)$

(C) fg, i.e., the function $x \mapsto f(x)g(x)$

(D) f/g, i.e., the function $x \mapsto f(x)/g(x)$

(E) $f \circ g$, i.e., the function $x \mapsto f(g(x))$

Answer: Option (D)

Explanation: See the example for the next question.

Performance review: 19 out of 24 got this. 4 chose (E), 1 chose (C).

Historical note (Math 153): 37 out of 42 got this. 4 chose (E), 1 chose (A).

(14) Which of the following is an example of a globally analytic function whose reciprocal is in $C^\infty(\mathbb{R})$ but is not globally analytic? *Two years ago:* 10/26 *correct*

(A) x
(B) x^2
(C) $x+1$
(D) x^2+1
(E) e^x

Answer: Option (D)

Explanation: The reciprocal $1/(x^2+1)$ is a rational function all of whose derivatives are rational functions with denominator a power of x^2+1, hence defined and continuous derivatives. Hence it is is $C^\infty(\mathbb{R})$. Further, the power series expansion for it is like a geometric series, which has radius of convergence 1, hence it is not globally analytic.

Why the other options are wrong:

Options (A) and (B): The reciprocals are not in $C^\infty(\mathbb{R})$ because the functions $1/x$ and $1/x^2$ are not defined or continuous at 0.

Option (C): The reciprocal is not in $C^\infty(\mathbb{R})$ because the function $1/(x+1)$ is not defined or continuous at -1.

Option (E): The reciprocal, which is $\exp(-x)$, is globally analytic.

Performance review: 22 out of 24 got this. 2 chose (E).

Historical note (Math 153): 28 out of 42 got this. 8 chose (C), 4 chose (E), 2 chose (A).

Historical note (last year): 9 out of 11 got this correct. 1 each chose (A) and (E).

Historical note (two years ago): 10 out of 26 people got this correct. 5 chose (C), 4 each chose (A) and (E), 2 chose (B), 1 left the question blank.

(15) Consider the rational function $1/\prod_{i=1}^{n}(x-\alpha_i)$, where the α_i are all distinct real numbers. This rational function is analytic about any point other than the α_is, and in particular its Taylor series converges to it on the interval of convergence. What is the radius of convergence for the Taylor series of the rational function about a point c not equal to any of the α_is? *Two years ago:* 10/26 *correct*

(A) It is the minimum of the distances from c to the α_is.
(B) It is the second smallest of the distances from c to the α_is.
(C) It is the arithmetic mean of the distances from c to the α_is.
(D) It is the second largest of the distances from c to the α_is.
(E) It is the maximum of the distances from c to the α_is.

Answer: Option (A)

Explanation: Since the Taylor series converges to the function on its interval of convergence, the interval of convergence must be contained in the domain of definition. In particular, it must exclude all the α_is. Hence, the radius of convergence cannot be more than the minimum of the distances from c to the α_is.

That it is exactly equal to the minimum can be shown by using the fact that we get a product of geometric series.

Performance review: 21 out of 24 got this. 2 chose (E), 1 chose (C).

Historical note (Math 153): 34 out of 42 got this. 6 chose (C), 2 chose (D).

Historical note (last year): 3 out of 11 got this correct. 4 chose (E), 2 chose (C), 1 chose (B).

Historical note (two years ago): 10 out of 26 people got this correct. 6 people chose (C), 5 chose (E), 2 each chose (B) and (D), 1 left the question blank.

(16) What is the interval of convergence of the Taylor series for arctan about 0? *Two years ago:* 11/26 *correct*

(A) $(-1,1)$
(B) $[-1,1)$
(C) $(-1,1]$
(D) $[-1,1]$
(E) All of $\mathbb{R}$

Answer: Option (D)

Explanation: For the boundary points, we use the alternating series theorem. See a more detailed discussion in the lecture notes.

Performance review: 19 out of 24 got this. 3 chose (A), 2 chose (E).

Historical note (Math 153): 40 out of 42 got this. 1 each chose (A) and (E).

Historical note (last year): 10 out of 11 got this correct. 1 chose (A).

Historical note (two years ago): 11 out of 26 people got this correct. 8 chose (A), 4 chose (C), 2 chose (B), 1 chose (E).

(17) What is the radius of convergence of the power series $\sum_{k=0}^{\infty} 2^{\sqrt{k}} x^k$? Please keep in mind the square root in the exponent.

(A) 0
(B) 1/2
(C) $1/\sqrt{2}$
(D) 1
(E) infinite

Answer: Option (D)

Explanation: The coefficients have subexponential growth, so the radius of convergence is 1.

Performance review: 22 out of 24 got this. 1 each chose (A) and (C).

Historical note (Math 153): 39 out of 42 got this. 3 chose (B).

TAKE-HOME CLASS QUIZ SOLUTIONS: DUE FRIDAY MARCH 1: MAX-MIN VALUES: ONE-VARIABLE RECALL

MATH 195, SECTION 59 (VIPUL NAIK)

1. Performance review

25 people took this 6-question quiz. The score distribution was as follows:

- Score of 2: 2 people
- Score of 3: 4 people
- Score of 4: 7 people
- Score of 5: 8 people
- Score of 6: 4 people

The question wise answers and performance review were as follows:

(1) Option (D): 18 people
(2) Option (B): 10 people
(3) Option (B): 25 people
(4) Option (B): 15 people
(5) Option (D): 24 people
(6) Option (A): 16 people

2. Solutions

(1) Suppose f is a function defined on a closed interval $[a, c]$. Suppose that the left-hand derivative of f at c exists and equals ℓ. Which of the following implications is **true in general**?
   (A) If $f(x) < f(c)$ for all $a \le x < c$, then $\ell < 0$.
   (B) If $f(x) \le f(c)$ for all $a \le x < c$, then $\ell \le 0$.
   (C) If $f(x) < f(c)$ for all $a \le x < c$, then $\ell > 0$.
   (D) If $f(x) \le f(c)$ for all $a \le x < c$, then $\ell \ge 0$.
   (E) None of the above is true in general.

   Answer: Option (D)

   Explanation: If $f(x) \le f(c)$ for all $a \le x < c$, then all difference quotients from the left are nonnegative. The limiting value, which is the left-hand derivative, is thus also nonnegative. See the lecture notes for more details.

   The other choices: Options (A) and (B) predict the wrong sign. Option (C) is incorrect because even though the difference quotients are all strictly positive, their limiting value could be 0. For instance, $\sin x$ on $[0, \pi/2]$ or x^3 on $[-1, 0]$.

   Performance review: 18 out of 25 got this. 4 chose (E), 3 chose (C).

   Historical note 1: 10 out of 18 people got this. 4 chose (E), 3 chose (B), 1 chose (C).

   Historical note 2: This question appeared in a 152 quiz, which 15 people took. At the time, 8 people got this correct. 5 people chose option (B) and 2 people chose option (E). It is likely that the people who chose option (B) made a sign computation error.

(2) Suppose f is a continuous function defined on an open interval (a, b) and c is a point in (a, b). Which of the following implications is **true**?
   (A) If c is a point of local minimum for f, then there is a value $\delta > 0$ and an open interval $(c - \delta, c + \delta) \subseteq (a, b)$ such that f is non-increasing on $(c - \delta, c)$ and non-decreasing on $(c, c + \delta)$.
   (B) If there is a value $\delta > 0$ and an open interval $(c - \delta, c + \delta) \subseteq (a, b)$ such that f is non-increasing on $(c - \delta, c)$ and non-decreasing on $(c, c + \delta)$, then c is a point of local minimum for f.

(C) If c is a point of local minimum for f, then there is a value $\delta > 0$ and an open interval $(c-\delta, c+\delta) \subseteq (a,b)$ such that f is non-decreasing on $(c-\delta, c)$ and non-increasing on $(c, c+\delta)$.
(D) If there is a value $\delta > 0$ and an open interval $(c-\delta, c+\delta) \subseteq (a,b)$ such that f is non-decreasing on $(c-\delta, c)$ and non-increasing on $(c, c+\delta)$, then c is a point of local minimum for f.
(E) All of the above are true.

Answer: Option (B).

Explanation: Since f is continuous, being non-increasing on $(c-\delta, c)$ implies being non-increasing on $(c-\delta, c]$. Similarly on the right side. In particular, this means that $f(c) \le f(x)$ for all $x \in (c-\delta, c+\delta)$, establishing c as a point of local minimum.

Performance review: 10 out of 25 got this. 12 chose (A), 2 chose (E), 1 chose (D).

Historical note 1: 7 out of 18 people got this correct. 5 chose (A), 3 each chose (C) and (D).

The other choices: Options (C) and (D) have the wrong kind of increase/decrease. Option (A) is wrong, though counterexamples are hard to come by. The reason Option (A) is wrong is the core of the reason that the first-derivative test does not always work: the function could be oscillatory very close to the point c, so that even though c is a point of local minimum, the function does not steadily become non-increasing to the left of c. The example discussed in the lecture notes is $|x|(2+\sin(1/x))$.

Historical note 2: This question appeared in a Math 152 quiz. At the time, 5 out of 15 people got this correct. 5 people chose (A), which is the converse of the statement. 2 people chose (D) and 1 person each chose (C) and (E). Thus, most people got the sign/direction part correct but messed up on which way the implication goes.

(3) Consider all the rectangles with perimeter equal to a fixed length $p > 0$. Which of the following **is true** for the unique rectangle which is a square, compared to the other rectangles?
(A) It has the largest area and the largest length of diagonal.
(B) It has the largest area and the smallest length of diagonal.
(C) It has the smallest area and the largest length of diagonal.
(D) It has the smallest area and the smallest length of diagonal.
(E) None of the above.

Answer: Option (B)

Explanation: We can see this easily by doing calculus, but it can also be deduced purely by thinking about how a square and a long thin rectangle of the same perimeter compare in terms of area and diagonal length.

Performance review: All 25 got this.

Historical note 1: 13 out of 18 people got this correct. 3 chose (E) and 2 chose (A).

Historical note 2: This question appeared in a past 152 quiz, and everybody got this correct.

Historical note3: This question appeared in an earlier 151 final, and 31 out of 33 people got it correct.

(4) Suppose the total perimeter of a square and an equilateral triangle is L. (We can choose to allocate all of L to the square, in which case the equilateral triangle has side zero, and we can choose to allocate all of L to the equilateral triangle, in which case the square has side zero). Which of the following statements **is true** for the sum of the areas of the square and the equilateral triangle? (The area of an equilateral triangle is $\sqrt{3}/4$ times the square of the length of its side).
(A) The sum is minimum when all of L is allocated to the square.
(B) The sum is maximum when all of L is allocated to the square.
(C) The sum is minimum when all of L is allocated to the equilateral triangle.
(D) The sum is maximum when all of L is allocated to the equilateral triangle.
(E) None of the above.

Answer: Option (B)

Quick explanation: The problem can also be solved using the rough heuristic that works for these kinds of problems: the maximum occurs when everything is allocated to the most efficient use, but the minimum typically occurs somewhere in between.

Full explanation: Suppose x is the part allocated to the square. Then $L - x$ is the part allocated to the equilateral triangle. The total area is:

$$A(x) = x^2/16 + (\sqrt{3}/4)(L-x)^2/9$$

Differentiating, we obtain:

$$A'(x) = \frac{x}{8} - \frac{\sqrt{3}}{18}(L-x) = x\left(\frac{1}{8} + \frac{\sqrt{3}}{18}\right) - \frac{\sqrt{3}}{18}L$$

We see that $A'(x) = 0$ at

$$x = \frac{L(\sqrt{3}/18)}{(1/8) + (\sqrt{3}/18)}$$

This number is indeed within the range of possible values of x.

Further, $A'(x) > 0$ for x greater than this and $A'(x) < 0$ for x less than this. Thus, this point is a local minimum and the maximum must occur at one of the endpoints. We plug in $x = 0$ to get $(\sqrt{3}/36)L^2$ and we plug in $x = L$ to get $L^2/16$. Since $1/16 > \sqrt{3}/36$, we obtain the the maximum occurs when $x = L$, which means that all the perimeter goes to the square.

Performance review: 15 out of 25 got this. 6 chose (C), 2 each chose (D) and (E).

Historical note 1: 7 out of 18 people got this correct. 7 chose (E), 2 each chose (C) and (D).

Historical note 2: This question appeared in a Math 152 quiz, and at the time 9 out of 15 people got this correct. 2 people each chose (E) and (C), 1 person chose (D), and 1 person left the question blank.

Historical note 3: This question appeared in a 152 midterm last year, and 20 of 29 people got it correct. In that midterm, option (E) wasn't there, so things became a little easier.

(5) Suppose x and y are positive numbers such as $x + y = 12$. For **what values** of x and y is x^2y maximum?
(A) $x = 3$, $y = 9$
(B) $x = 4$, $y = 8$
(C) $x = 6$, $y = 6$
(D) $x = 8$, $y = 4$
(E) $x = 9$, $y = 3$

Answer: Option (D).

Quick explanation: This is a special case of the general Cobb-Douglas situation where we want to maximize $x^a(C-x)^b$. The general solution is to take $x = Ca/(a+b)$, i.e., to take x and $C - x$ in the proportion of a to b.

Full explanation: We need to maximize $f(x) := x^2(12-x)$, subject to $0 < x < 12$. Differentiating, we get $f'(x) = 3x(8-x)$, so 8 is a critical point. Further, we see that f' is positive on $(0, 8)$ and negative on $(8, 12)$, so f attains its maximum (in the interval $(0, 12)$) at 8.

Performance review: 24 out of 25 got this. 1 chose (E).

Historical note 1: 11 out of 18 people got this correct. 5 chose (E), 2 chose (C).

Historical note 2: This question appeared in a Math 152 quiz. At the time, 12 out of 15 people got this correct. 2 people chose (E) and 1 person chose (B). Of the people who got this correct, some seem to have computed the numerical values and others seem to have used calculus. Some who did not show any work may have used the general result of the Cobb-Douglas situation.

(6) Consider the function $p(x) := x^2 + bx + c$, with x restricted to integer inputs. Suppose b and c are integers. The minimum value of p is attained either at a single integer or at two consecutive integers. Which of the following is a **sufficient condition** for the minimum to occur at two consecutive integers?
(A) b is odd
(B) b is even
(C) c is odd
(D) c is even
(E) None of these conditions is sufficient.

Answer: Option (A)

Explanation: The graph of f is symmetric about the half-integer axis value $-b/2$. It is an upward-facing parabola. For odd b, it attains its minimum among integers at the two consecutive integers $-b/2+1/2$ and $-b/2-1/2$. When b is even, the minimum is attained uniquely at $-b/2$, which is itself an integer. c being odd or even tells us nothing.

Performance review: 16 out of 25 got this. 4 chose (E), 2 each chose (B) and (D), 1 chose (C).

Historical note 1: 2 out of 18 people got this correct. 8 chose (E), 5 chose (B), 3 chose (C).

Historical note 2: This question appeared in a Math 152 quiz. At the time, 4 out of 15 people got this correct. 8 people chose (E) and 3 people chose (B).

TAKE-HOME CLASS QUIZ SOLUTIONS: DUE MONDAY MARCH 11 (POSTPONED TO MARCH 13): MAX-MIN VALUES: TWO-VARIABLE VERSION

MATH 195, SECTION 59 (VIPUL NAIK)

1. Performance review

25 people took this 7-question quiz. The score distribution was as follows:

- Score of 1: 2 people.
- Score of 2: 1 person.
- Score of 3: 6 people.
- Score of 4: 4 people.
- Score of 5: 7 people.
- Score of 6: 5 people.

The question wise answers and performance information are below:

(1) Option (B): 23 people.
(2) Option (B): 14 people.
(3) Option (C): 3 people.
(4) Option (C): 19 people.
(5) Option (D): 7 people.
(6) Option (E): 21 people.
(7) Option (E): 16 people.

2. Solutions

(1) Suppose $F(x, y) := f(x) + g(y)$, i.e., F is additively separable. Suppose f and g are differentiable functions of one variable, defined for all real numbers. What can we say about the critical points of F in its domain $\mathbb{R}^2$?
   (A) F has a critical point at (x_0, y_0) iff x_0 is a critical point for f *or* y_0 is a critical point for g.
   (B) F has a critical point at (x_0, y_0) iff x_0 is a critical point for f *and* y_0 is a critical point for g.
   (C) F has a critical point at (x_0, y_0) iff $x_0 + y_0$ is a critical point for $f + g$, i.e., the function $x \mapsto f(x) + g(x)$.
   (D) F has a critical point at (x_0, y_0) iff x_0y_0 is a critical point for fg, i.e., the function $x \mapsto f(x)g(x)$.
   (E) None of the above.

   Answer: Option (B)

   Explanation: $F_x(x_0, y_0) = f'(x_0)$ and $F_y(x_0, y_0) = g'(y_0)$. In order for both of these to be 0, we must have $f'(x_0) = g'(y_0) = 0$. Thus, x_0 is a critical point for f and y_0 is a critical point for g.

   Performance review: 23 out of 25 people got this. 2 chose (C).

   Historical note (last time): 12 out of 17 people got this correct. 3 chose (C) and 2 chose (A).

(2) Suppose $F(x, y) := f(x)g(y)$ is a multiplicatively separable function. Suppose f and g are both differentiable functions of one variable defined for all real inputs. Consider a point (x_0, y_0) in the domain of F, which is $\mathbb{R}^2$. Which of the following is true?
   (A) F has a critical point at (x_0, y_0) if and only if x_0 is a critical point for f and y_0 is a critical point for g.
   (B) If x_0 is a critical point for f and y_0 is a critical point for g, then (x_0, y_0) is a critical point for F. However, the converse is not necessarily true, i.e., (x_0, y_0) may be a critical point for F even without x_0 being a critical point for f and y_0 being a critical point for g.
   (C) If (x_0, y_0) is a critical point for F, then x_0 must be a critical point for f and y_0 must be a critical point for g. However, the converse is not necessarily true.

(D) (x_0, y_0) is a critical point for F if and only if *at least* one of these is true: x_0 is a critical point for f and y_0 is a critical point for g.
(E) None of the above.

Answer: Option (B)

Explanation: We have $F_x(x_0, y_0) = f'(x_0)g(y_0)$ and $F_y(x_0, y_0) = f(x_0)g'(y_0)$. We see that if $f'(x_0) = 0$ and $g'(y_0) = 0$, then F has a critical point at (x_0, y_0). However, F could also have a critical point for other reasons: for instance, $f(x_0) = f'(x_0) = 0$ (and other cases).

Performance review: 14 out of 25 got this. 4 chose (D), 3 each chose (A) and (C), 1 chose (E).

Historical note (last time): 9 out of 17 people got this correct. 5 chose (A), 2 chose (D), 1 chose (C).

(3) Consider a homogeneous polynomial $ax^2 + bxy + cy^2$ of degree two in two variables x and y. Assume that at least one of the numbers a, b, and c is nonzero. What can we say about the local extreme values of this polynomial on $\mathbb{R}^2$?
(A) If $b^2 - 4ac < 0$, then the function has no local extreme values and its value is unbounded from both above and below. If $b^2 - 4ac = 0$, the function has local extreme value 0 and this is attained on a line through the origin. If $b^2 - 4ac > 0$, the function has local extreme value 0 and this is attained only at the origin.
(B) If $b^2 - 4ac < 0$, then the function has no local extreme values and its value is unbounded from both above and below. If $b^2 - 4ac > 0$, the function has local extreme value 0 and this is attained on a line through the origin. If $b^2 - 4ac = 0$, the function has local extreme value 0 and this is attained only at the origin.
(C) If $b^2 - 4ac > 0$, then the function has no local extreme values and its value is unbounded from both above and below. If $b^2 - 4ac = 0$, the function has local extreme value 0 and this is attained on a line through the origin. If $b^2 - 4ac < 0$, the function has local extreme value 0 and this is attained only at the origin.
(D) If $b^2 - 4ac > 0$, then the function has no local extreme values and its value is unbounded from both above and below. If $b^2 - 4ac < 0$, the function has local extreme value 0 and this is attained on a line through the origin. If $b^2 - 4ac = 0$, the function has local extreme value 0 and this is attained only at the origin.
(E) If $b^2 - 4ac = 0$, then the function has no local extreme values and its value is unbounded from both above and below. If $b^2 - 4ac < 0$, the function has local extreme value 0 and this is attained on a line through the origin. If $b^2 - 4ac > 0$, the function has local extreme value 0 and this is attained only at the origin.

Answer: Option (C)

Explanation: See discussion of homogeneous quadratic in the relevant lecture notes.

Performance review: 3 out of 25 got this. 10 chose (B), 7 chose (D), 4 chose (A), 1 chose (E).

Historical note (last time): 5 out of 17 people got this correct. 4 chose (A), 4 chose (B), 2 chose (D), 2 chose (E).

A subset of $\mathbb{R}^n$ is termed *convex* if the line segment joining any two points in the subset is completely within the subset. A function f of two variables defined on a closed convex domain is termed *quasiconvex* if given any two points P and Q in the domain, the maximum of f restricted to the line segment joining P and Q is attained at one (possibly both) of the endpoints P or Q.

There are many examples of quasiconvex functions, including linear functions (which are quasiconvex but not strictly quasiconvex) and all convex functions.

(4) What can we say about the maximum of a continuous quasiconvex function defined on the circular disk $x^2 + y^2 \leq 1$?
(A) It must be attained at the center of the disk, i.e., the origin $(0, 0)$.
(B) It must be attained somewhere in the interior of the disk, but we cannot be more specific with the given information.
(C) It must be attained somewhere on the boundary circle $x^2 + y^2 = 1$. However, we cannot be more specific than that with the given information.
(D) It must be attained at one of the four points $(1, 0)$, $(0, 1)$, $(-1, 0)$, and $(0, -1)$.
(E) It could be attained at any point. We cannot be specific at all.

Answer: Option (C)

Explanation: For any point in the interior of the circular disk, the point can be put on a chord with endpoints on the boundary circle. The maximum of the function restricted to this chord occurs at one of the boundary points, hence, the value at any point in the interior is equaled or exceeded by some point in the boundary. Thus, the maximum is attained at some point in the boundary.

However, no point in the boundary can be put in the interior of a line segment joining two other points, i.e., each point in the boundary is extreme. So we cannot narrow things down further.

Performance review: 19 out of 25 got this. 3 chose (D). 1 each chose (A), (B), and (E).

Historical note (last time): 8 out of 17 people got this correct. 5 chose (B), 3 chose (E), 1 chose (D).

(5) What can we say about the maximum of a continuous quasiconvex function defined on the square region $|x| + |y| \leq 1$? This is the region bounded by the square with vertices $(1, 0)$, $(0, 1)$, $(-1, 0)$, and $(0, -1)$.
   (A) It must be attained at the center of the square, i.e., the origin $(0, 0)$.
   (B) It must be attained somewhere in the interior of the square, but we cannot be more specific with the given information.
   (C) It must be attained somewhere on the boundary square $|x| + |y| \leq 1$. However, we cannot be more specific than that with the given information.
   (D) It must be attained at one of the four points $(1, 0)$, $(0, 1)$, $(-1, 0)$, and $(0, -1)$.
   (E) It could be attained at any point. We cannot be specific at all.

   Answer: Option (D)

   Explanation: We can first push to the boundary the same way as for the circular disk. Now, unlike the circular disk, we note that any point in the boundary that is to one of the vertices is on a line segment joining two adjacent vertices, so we can push out to the vertices.

   Performance review: 13 out of 25 got this. 7 chose (D), 3 chose (B), 1 eacch chose (A) and (E).

   Historical note (last time): 3 out of 17 people got this correct. 8 chose (C), 3 chose (B), 2 chose (A), 1 chose (E).

(6) Suppose $F(x, y) := f(x) + g(y)$, i.e., F is additively separable. Suppose f and g are continuous functions of one variable, defined for all real numbers. Which of the following statements about local extrema of F is **false**?
   (A) If f has a local minimum at x_0 and g has a local minimum at y_0, then F has a local minimum at (x_0, y_0).
   (B) If f has a local minimum at x_0 and g has a local maximum at y_0, then F has a saddle point at (x_0, y_0).
   (C) If f has a local maximum at x_0 and g has a local minimum at y_0, then F has a saddle point at (x_0, y_0).
   (D) If f has a local maximum at x_0 and g has a local maximum at y_0, then F has a local maximum at (x_0, y_0).
   (E) None of the above, i.e., they are all true.

   Answer: Option (E)

   Explanation: This is discussed in the lecture notes.

   Performance review: 21 out of 25 got this. 2 chose (B). 1 each chose (A) and (D).

(7) Suppose $F(x, y) := f(x)g(y)$ is a multiplicatively separable function. Suppose f and g are both continuous functions of one variable defined for all real inputs. Consider a point (x_0, y_0) in the domain of F, which is $\mathbb{R}^2$. Which of the following statements about local extrema is **true**?
   (A) If f has a local minimum at x_0 and g has a local minimum at y_0, then F has a local minimum at (x_0, y_0).
   (B) If f has a local minimum at x_0 and g has a local maximum at y_0, then F has a saddle point at (x_0, y_0).
   (C) If f has a local maximum at x_0 and g has a local minimum at y_0, then F has a saddle point at (x_0, y_0).
   (D) If f has a local maximum at x_0 and g has a local maximum at y_0, then F has a local maximum at (x_0, y_0).

(E) None of the above, i.e., they are all false.

Answer: Option (E)

Explanation: We also need information about the signs of $f(x_0)$ and $g(y_0)$ since that affects the conclusion. All the options above would be *true* (rather than false) if both $f(x_0)$ and $g(y_0)$ are known to be positive.

Performance review: 16 out of 25 got this. 3 chose (A), 2 each chose (B), (C), and (D).