THREE-DIMENSIONAL GEOMETRY

MATH 195, SECTION 59 (VIPUL NAIK)

Corresponding material in the book: Section 12.1

What students should definitely get: The use of three coordinates to describe points in space. Right-hand rule and orientation issues. Octants and coordinate planes. Distance formula. Equation of sphere.

What students should hopefully get: Three-dimensionality. Relation between describing equations and dimensionality (dimensionality of surfaces, curves). The special case of the linear situation.

Executive summary

Words ...

(1) Three-dimensional space is coordinatized using a Cartesian coordinate system by selecting three mutually perpendicular axes passing through a point called the origin: the x-axis, y-axis, and z-axis. These satisfy the right-hand rule. The coordinates of a point are written as a 3-tuple (x, y, z).

(2) There are $2^3 = 8$ octants based on the signs of each of the coordinates. There are three coordinate planes, each corresponding to the remaining coordinate being zero (the xy-plane corresponds to $z = 0$, etc.). There are three axes, each corresponding to the other two coordinates being zero (e.g., the x-axis corresponds to $y = z = 0$).

(3) The distance formula between points with coordinates (x_1, y_1, z_1) and (x_2, y_2, z_2) is:

$$\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

This is similar to the formula in two dimensions and the squares and square root arises from the Pythagorean theorem.

(4) The equation of a sphere with center having coordinates (h, k, l) and radius r is $(x-h)^2 + (y-k)^2 + (z-l)^2 = r^2$. Given an equation, we can try completing the square to see if it fits the model for the equation of a sphere.

1. Quick introduction

1.1. What's up with the number three?

Why is three-dimensional space considered *three*-dimensional? Because, in order to describe a point in this space, it requires three coordinates, or three independent pieces of information. Remember, dimensionality is the *number of degrees of freedom*, or the *number of freely varying parameters*.

For now, the focus is on describing the behavior of a *Cartesian* three-dimensional coordinate system. However, it is important to note that *any* successful coordinate system to describe three-dimensional space must use three freely varying parameters. There does exist (though we will not be talking about it right now) a *polar* coordinate system in three dimensions.

1.2. Three coordinates: x, y, and z.

The three-dimensional Cartesian coordinate system involves the following: choice of a point called the *origin*, and three mutually perpendicular directed lines called the x-axis, y-axis, and z-axis respectively. To specify a point with three coordinates, we move parallel to the x-axis by the value of the x-coordinate, then parallel to the y-axis by the value of the y-coordinate, then parallel to the z-axis by the value of the z-coordinate.

The way to denote/describe a point with x-coordate x_0, y-coordinate y_0, and z-coordinate z_0 is as (x_0, y_0, z_0). Or, more generally, we just write (x, y, z).

1.3. **Orientation issues.** Recall that, for the two-dimensional plane, we adopt an orientation convention, namely, the convention that the shorter angle from the positive x-axis to the positive y-axis (measured $\pi/2$) be counter-clockwise. In a three-dimesional system, we must adopt a similar orientation convention. This convention is somewhat artificial and in fact if you were doing mathematics properly then such a convention would be meaningless. But since we're trying to do mathematics the shortcut way, we will need to introduce this convention.

The convention is called the *right-hand rule.* This says that if you grip the z-axis with your right hand and curl the fingers of your right hand so as to move from the x-axis to the y-axis, and make your thumb point along the z-axis away from the fingers, then it points along the positive z-axis. The book has a nice picture of this, which I will not reproduce here.

We will thus deal with *right-handed coordinate systems.* The mirror reflection of a right-hand coordinate system is a left-handed coordinate system. There isn't really any difference between left-handed and right-handed coordinate systems as far as their geometry is concerned. However, they are different in the sense that no amount of rotation can turn a right-handed coordinate system into a left-handed coordinate system.

1.4. **Octants, axes, planes.** Since there are three coordinates, there are a total of $2^3 = 8$ possible sign combinations for the coordinates. These eight sign combination give rise to eight chambers of three-dimensional space, and these chambers are called *octants.* They are the three-dimensional analogues of the *quadrants* in two-dimensional space, of which there are $2^2 = 4$.

The eight octants are:

- Positive x, positive y, positive z
- Positive x, positive y, negative z
- Positive x, negative y, positive z
- Positive x, negative y, negative z
- Negative x, positive y, positive z
- Negative x, positive y, negative z
- Negative x, negative y, positive z
- Negative x, negative y, negative z

There are also cases where one or more of the coordinates takes the value 0. These are discussed below:

- The x-coordinate takes the value 0: The yz-plane.
- The y-coordinate takes the value 0: The xz-plane.
- The z-coordinate takes the value 0: The xy-plane.
- Both the y and z coordinates take the value 0: The x-axis.
- Both the x and z coordinates take the value 0: The y-axis.
- Both the y and z coordinates take the value 0: The z-axis.
- All three coordinates are zero: We get the origin.

1.5. **The distance formula.** The distance between the points (x_1, y_1, z_1) and (x_2, y_2, z_2) is given by the formula:

$$\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

Note that this is very similar to the two-dimensional distance formula, and has a similar justification. Please note that the *appearance of squares and square roots is independent of dimensionality.* It has to do with the Pythagorean theorem. So when we are working in three dimensions, we still deal with squares and square roots, not with cubes and cube roots.

2. Curves and surfaces

2.1. **The arithmetic of dimensionality of curves and surfaces.** A surface is something two-dimensional, a curve is something one-dimensional, and a point is something zero-dimensional.

Every condition, relation, or equation creates a constraint and reduces the dimensionality of the set of possibilities by one. In particular:

- A single equation reduces the dimensionality of the solution space by one. Since $3 - 1 = 2$, the solution set to a single equation in three dimensions is something two-dimensional, such as a plane or surface, or a union of finitely many planes and surfaces.
- A pair of two equations reduces the dimensionality of the solution space by two. Since $3 - 2 = 1$, the solution set to a pair of such equations is something one-dimensional, such as a line or curve, or a union of finitely many lines and curves.
- A triple of equations reduces the dimensionality of the solution space by three. Since $3 - 3 = 0$, the solution set to a tripe of such equations is something zero-dimensional, such as a point or a finite collection of points.

2.2. **Linear equations and systems.** We will talk about these in considerably more detail later on, but here we simply note how linear equations fit into the model above.

- The solution set to a *linear equation* in x, y, and z is a plane, which is a flat example of a surface.

  In particular, the solution set to an equation of the form $x = x_0$ is a plane parallel to the yz-plane. Similarly, $y = y_0$ gives a plane parallel to the xz-plane, and $z = z_0$ gives a plane parallel to the xy-plane.
- The solution set to a *consistent but inequivalent pair of linear equations* in x, y, and z is a line which is a flat example of a curve.

  In particular, the solution set to a pair of equations of the form $x = x_0$, $y = y_0$ is a line parallel to the z-axis. Similarly with the coordinate roles interchanged.
- The solution set to a system of three consistent but independent linear equations in x, y and z is a single point.

2.3. **Ignoring one coordinate.** Suppose we have a relation $R(x, y) = 0$. The solution space to this in *three* dimensions is the surface obtained by taking the solution set in the xy-plane and then translating it along the z-direction to cover every possible z-coordinate. We can think of it as something cylinder-like.

For instance, the relation $x^2 + y^2 = 25$ gives a circle of radius 5 centered at the origin when viewed purely in the xy-plane. The solution set overall is a cylinder stretching infinitely in both directions, whose axis is the z-axis and whose cross sections are these circles.

Similar remarks hold for a relation purely in terms of y and z or a relation purely in terms of x and z.

2.4. **Equation of a sphere.** The equation of a sphere whose center has coordinates (h, k, l) and whose radius is r is given by:

$$(x - h)^2 + (y - k)^2 + (z - l)^2 = r^2$$

Given an equation, we can try doing things like completing the square and see if, after we do that, we get the equation of a sphere.