TAKE-HOME CLASS QUIZ: INTEGRATION BY PARTS: DUE MONDAY NOVEMBER 26

MATH 153, SECTION 59 (VIPUL NAIK)

Your name (print clearly in capital letters): ______________________________________

YOU ARE FREE TO DISCUSS ALL QUESTIONS, BUT PLEASE MAKE SURE TO ONLY ENTER ANSWER CHOICES THAT YOU PERSONALLY ENDORSE

In the questions below, we say that a function is *expressible in terms of elementary functions* or *elementarily expressible* if it can be expressed in terms of polynomial functions, rational functions, radicals, exponents, logarithms, trigonometric functions and inverse trigonometric functions using pointwise combinations, compositions, and piecewise definitions. We say that a function is *elementarily integrable* if it has an elementarily expressible antiderivative.

Note that if a function is elementarily expressible, so is its derivative on the domain of definition.

We say that a function f is k times elementarily integrable if there is an elementarily expressible function g such that f is the k^{th} derivative of g.

We say that the integrals of two functions are *equivalent up to elementary functions* if an antiderivative for one function can be expressed using an antiderivative for the other function and elementary function, again piecing them together using pointwise combination, composition, and piecewise definitions.

(1) Suppose F, G are continuously differentiable functions defined on all of $\mathbb{R}$. Suppose a, b are real numbers with $a < b$. Suppose, further, that $G(x)$ is identically zero everywhere except on the open interval (a, b). Then, what can we say about the relationship between the numbers $P = \int_a^b F(x)G'(x)\,dx$ and $Q = \int_a^b F'(x)G(x)\,dx$?
(A) $P = Q$
(B) $P = -Q$
(C) $PQ = 0$
(D) $P = 1 - Q$
(E) $PQ = 1$

Your answer: ____________________________

(2) Consider the integration $\int p(x)q''(x)\,dx$. Apply integration by parts twice, first taking p as the part to differentiate, and q as the part to integrate, and then again apply integration by parts to avoid a circular trap. What can we conclude?
(A) $\int p(x)q''(x)\,dx = \int p''(x)q(x)\,dx$
(B) $\int p(x)q''(x)\,dx = \int p'(x)q'(x)\,dx - \int p''(x)q(x)\,dx$
(C) $\int p(x)q''(x)\,dx = p'(x)q'(x) - \int p''(x)q(x)\,dx$
(D) $\int p(x)q''(x)\,dx = p(x)q'(x) - p'(x)q(x) + \int p''(x)q(x)\,dx$
(E) $\int p(x)q''(x)\,dx = p(x)q'(x) - p'(x)q(x) - \int p''(x)q(x)\,dx$

Your answer: ____________________________

(3) Suppose p is a polynomial function. In order to find the indefinite integral for a function of the form $x \mapsto p(x)\exp(x)$, the general strategy, which always works, is to take $p(x)$ as the part to differentiate and $\exp(x)$ as the part to integrate, and keep repeating the process. Which of the following is the best explanation for why this strategy works?
(A) exp can be repeatedly differentiated (staying exp) and polynomials can be repeatedly integrated (giving polynomials all the way).
(B) exp can be repeatedly integrated (staying exp) and polynomials can be repeatedly differentiated, eventually becoming zero.

(C) exp and polynomials can both be repeatedly differentiated.
(D) exp and polynomials can both be repeatedly integrated.
(E) We need to use the recursive version of integration by parts whereby the original integrand reappears after a certain number of applications of integration by parts (i.e., the polynomial equals one of its higher derivatives, up to sign and scaling).

Your answer: ____________________________

(4) Consider the function $x \mapsto \exp(x)\sin x$. This function can be integrated using integration by parts. What can we say about how integration by parts works?
(A) We choose exp as the part to integrate and sin as the part to differentiate, and apply this process once to get the answer directly.
(B) We choose exp as the part to integrate and sin as the part to differentiate, and apply this process once, then use a *recursive* method (identify the integrals on the left and right side) to get the answer.
(C) We choose exp as the part to integrate and sin as the part to differentiate, and apply this process twice to get the answer directly.
(D) We choose exp as the part to integrate and sin as the part to differentiate, and apply this process twice, then use a *recursive* method (identify the integrals on the left and right side) to get the answer.
(E) We choose exp as the part to integrate and sin as the part to differentiate, and we apply integration by parts four times to get the answer directly.

Your answer: ____________________________

(5) Consider the statements P and Q, where P states that every rational function is elementarily integrable, and Q states that any rational function is k times elementarily integrable for all positive integers k.

Which of the following additional observations is **correct** and **allows us to deduce** Q given P? *Two years ago:* 18/27 *correct*
(A) There is no way of deducing Q from P because P is true and Q is false.
(B) The antiderivative of a rational function can always be chosen to be a rational function, hence Q follows from a repeated application of P.
(C) Using integration by parts, we see that repeated integration of a function f is equivalent to integrating f, f^2, f^3, and higher powers of f (the powers here are pointwise products, not compositions). If f is a rational function, each of these is also a rational function. Applying P, each of these is elementarily integrable, hence f is k times elementarily integrable for all k.
(D) Using integration by parts, we see that repeated integration of a function f is equivalent to integrating f, f', f'', and higher derivatives of f. If f is a rational function, each of these is also a rational function. Applying P, each of these is elementarily integrable, hence f is k times elementarily integrable for all k.
(E) Using integration by parts, we see that repeated integration of a function f is equivalent to integrating each of the functions $f(x)$, $xf(x)$, $\dots$. If f is a rational function, each of these is also a rational function. Applying P, each of these is elementarily integrable, hence f is k times elementarily integrable for all k.

Your answer: ____________________________

(6) Suppose f is a continuous function on all of $\mathbb{R}$ and is the third derivative of an elementarily expressible function, but is not the fourth derivative of any elementarily expressible function. In other words, f can be integrated three times but not four times within the collection of elementarily expressible functions. What is the **largest positive integer** k such that $x \mapsto x^k f(x)$ is *guaranteed to be* **elementarily integrable**? *Two years ago:* 23/27 *correct*
(A) 1
(B) 2

(C) 3
(D) 4
(E) 5

Your answer: ____________________________

(7) Suppose f is a continuous function on $(0, \infty)$ and is the third derivative of an elementarily expressible function, but is not the fourth derivative of any elementarily expressible function. In other words, f can be integrated three times but not four times within the collection of elementarily expressible functions. What is the **largest positive integer** k such that the function $x \mapsto f(x^{1/k})$ with domain $(0, \infty)$ is *guaranteed to be* **elementarily integrable**? *Two years ago:* 14/27 *correct*
(A) 1
(B) 2
(C) 3
(D) 4
(E) 5

Your answer: ____________________________

(8) Of these five functions, four of the functions are elementarily integrable and can be integrated using integration by parts. The other one function is **not elementarily integrable**. Identify this function. *Two years ago:* 22/27 *correct*
(A) $x \mapsto x \sin x$
(B) $x \mapsto x \cos x$
(C) $x \mapsto x \tan x$
(D) $x \mapsto x \sin^2 x$
(E) $x \mapsto x \tan^2 x$

Your answer: ____________________________

(9) Consider the four functions $f_1(x) = \sqrt{\sin x}$, $f_2(x) = \sin\sqrt{x}$, $f_3(x) = \sin^2 x$ and $f_4(x) = \sin(x^2)$, all viewed as functions on the interval $[0, 1]$ (so they are all well defined). Two of these functions are elementarily integrable; the other two are not. Which are **the two elementarily integrable functions**? *Two years ago:* 17/27 *correct*
(A) f_3 and f_4.
(B) f_1 and f_3.
(C) f_1 and f_4.
(D) f_2 and f_3.
(E) f_2 and f_4.

Your answer: ____________________________

(10) Suppose f is an elementarily expressible and infinitely differentiable function on the positive reals (so all derivatives of f are also elementarily expressible). An antiderivative for $f''(x)/x$ is **not equivalent** up to elementary functions to **which one** of the following? *Two years ago:* 10/27 *correct*
(A) An antiderivative for $x \mapsto f''(e^x)$, domain all of $\mathbb{R}$.
(B) An antiderivative for $x \mapsto f'(e^x/x)$, domain positive reals.
(C) An antiderivative for $x \mapsto f'''(x)(\ln x)$, domain positive reals.
(D) An antiderivative for $x \mapsto f'(1/x)$, domain positive reals.
(E) An antiderivative for $x \mapsto f(1/\sqrt{x})$, domain positive reals.

Your answer: ____________________________

(11) Of the five functions below, four of them have antiderivatives that are equivalent up to elementary functions, i.e., an antiderivative for any one of them can be used to provide an antiderivative for

the other three. The fifth function has an antiderivative that is **not equivalent** to any of these. Identify the fifth function. *Two years ago:* 7/27 *correct*

(A) $x \mapsto e^{e^x}$, domain all reals
(B) $x \mapsto \ln(\ln x)$, domain $(1, \infty)$
(C) $x \mapsto e^x/x$, domain $(0, \infty)$
(D) $x \mapsto 1/(\ln x)$, domain $(1, \infty)$
(E) $x \mapsto 1/(\ln(\ln x))$, domain (e, ∞)

Your answer: ____________________________

(12) Which of the following functions has an antiderivative that is **not equivalent** up to elementary functions to the antiderivative of $x \mapsto e^{-x^2}$? *Two years ago:* 10/27 *correct*

(A) $x \mapsto e^{-x^4}$
(B) $x \mapsto e^{-x^{2/3}}$
(C) $x \mapsto e^{-x^{2/5}}$
(D) $x \mapsto x^2 e^{-x^2}$
(E) $x \mapsto x^4 e^{-x^2}$

Your answer: ____________________________

(13) Which of the following has an antiderivative that is **not equivalent** up to elementary functions to the antiderivative of the function $f(x) := e^x/x, x > 0$? *Two years ago:* 5/27 *correct*

(A) $x \mapsto e^x/\sqrt{x}, x > 0$
(B) $x \mapsto e^x/x^2, x > 0$
(C) $x \mapsto e^x(\ln x), x > 0$
(D) $x \mapsto e^{1/\sqrt{x}}, x > 0$
(E) $x \mapsto e^{1/x}, x > 0$

Your answer: ____________________________