TAYLOR POLYNOMIALS AND TAYLOR SERIES

MATH 153, SECTION 55 (VIPUL NAIK)

Corresponding material in the book: Section 12.6, 12.7.

What students should definitely get: The definition of Taylor polynomial, the definition of Taylor series, the definition of remainder. The notion of convergence for a given value of x. The statement of Taylor's theorem, the Lagrange formula for remainder, and the max-estimate.

What students should hopefully get: The connection with earlier ideas of order of zero and approximation by polynomials.

Executive summary

0.1. Taylor series at 0. Words ...

(1) Suppose f is a function defined and n times differentiable at 0. Then, the n^{th} Taylor polynomial of f is:

$$P_n(x) = \sum_{k=0}^{n} f^{(k)}(0)\frac{x^k}{k!}$$

(2) The degree of the n^{th} Taylor polynomial is $\leq n$. Note that it is exactly n if and only if $f^{(n)}(0) \neq 0$.

(3) The number of nonzero terms in the n^{th} Taylor polynomial is at most $n+1$, but it could be substantially less, depending on how many of the $n+1$ numbers $f(0)$, $f'(0)$, $\dots$, $f^{(n)}(0)$ are nonzero.

(4) For $m < n$, the m^{th} Taylor polynomial is the truncation to terms of degree $\leq m$ of the n^{th} Taylor polynomial.

(5) The Taylor series is the infinite sum:

$$\sum_{k=0}^{\infty} f^{(k)}(0)\frac{x^k}{k!}$$

The Taylor polynomials are thus truncations of the Taylor series.

(6) The Taylor series for exp, sin, cos, sinh, and cosh are particularly easy to write down because the sequence of derivatives of these functions is periodic, hence so is the sequence of derivative values at 0. [Review the formulas]

(7) The Taylor series of a polynomial is the same polynomial.

(8) For exp, sin, cos, sinh, cosh, and polynomials, the Taylor series converges to the function everywhere.

(9) The Taylor series for an even function has nonzero coefficients only for even powers of x. In other words, the Taylor series for an even function is an even power series. Similarly, the Taylor series for an odd function has nonzero coefficients only for odd powers of x.

(10) The Taylor series of the derivative is the derivative (via term wise differentiation) of the Taylor series.

(11) The Taylor series operator is linear and multiplicative: the Taylor series for $f+g$ is the sum of the Taylor series for f and g, and the Taylor series for $f \cdot g$ is the product for the Taylor series for f and g. [Note: Multiplying two Taylor series is a pain in general. Howveer, if one of the functions is a polynomial, it is not too hard. For instance, xe^x, $x^2\sin(x^2)$, $(2x+1)\cos x$]

(12) Suppose g is a polynomial with zero constant term. Then, the Taylor series for $f \circ g$ can be obtained by taking the Taylor series for f, replacing x by $g(x)$ throughout, and simplifying. Consider, for instance, the Taylor series for $\sin(x^2)$ and $e^{-x^2/2}$.

(13) The n^{th} (Taylor) *remainder* R_n for a function f is defined as $f - P_n$, where P_n is the n^{th} Taylor polynomial for f. Taylor's theorem states that if f is at least $(n+1)$ times differentiable, the remainder R_n is given by $R_n(x) = \frac{1}{n!}\int_0^x f^{(n+1)}(t)(x-t)^n\,dt$.

(14) The Lagrange formula is a corollary of Taylor's theorem, and it states that there exists c between 0 and x such that $R_n(x) = f^{(n+1)}(c)x^{n+1}/(n+1)!$. Here, c between 0 and x means $c \in [0, x]$ if $x > 0$ and $c \in [x, 0]$ if $x < 0$.

(15) A further corollary of the Lagrange formula (that we call the max-estimate here) states that $|R_n(x)|$ is at most $|x|^{n+1}/(n+1)!$ times the maximum value of $|f^{(n+1)}(t)|$ for t between 0 and x.

(16) The max-estimate can be used to justify that the Taylor series for exp, sin, and cos actually converge to the respective functions. This is done by showing that for any $x \in \mathbb{R}$, we have $\lim_{n\to\infty} R_n(x) = 0$.

(17) The zeroth Taylor polynomial for a function f is the constant function $f(0)$. The first Taylor polynomial is the constant/linear function $f(0) + f'(0)x$. This describes the tangent line to the function, and is the *best straight line approximation* to the function locally around 0. More generally, the n^{th} Taylor polynomial is the best approximation to the function around 0 among the polnyomials of degree $\leq n$.

0.2. **Taylor series in $x - a$.** It is an instructive exercise (and I urge you to do this) to translate all the statements about Taylor series around 0 to the corresponding statements about Taylor series around an arbitrary $a \in \mathbb{R}$. In particular, see if you can correctly translate (pun intended) Taylor's theorem, the Lagrange formula, and its max-estimate corollary. We will go over this further in the review session.

1. Taylor polynomials: Definition and basic computation

1.1. **The definition of Taylor polynomial.** Suppose f is a function that is n times differentiable at 0. The n^{th} Taylor polynomial of f at 0, sometimes denoted P_n, is defined as:

$$P_n(x) := \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} x^k$$

Here, $f^{(k)}(0)$ denotes the k^{th} derivative of f at 0. In particular, $f^{(0)}(0)$ is simply the value of f at 0. Explicitly, we have:

$$P_n(x) = f(0) + f'(0)x + \frac{f''(0)x^2}{2} + \frac{f'''(0)x^3}{6} + \cdots + \frac{f^{(n)}(0)x^n}{n!}$$

Note that unlike the typical way we write polynomials with the highest powers first, Taylor polynomials are typically written with the lowest powers first. There are good reasons for this, which will become clearer later. For now, the main justification that you can remember for this is that the Taylor polynomial P_n is evaluated for x close to 0. For such x, it is the smaller powers of x that are numerically larger.

Also note that if $m < n$, then the Taylor polynomial P_m for a function f is an initial segment of the Taylor polynomial P_n for f.

1.2. **The definition of Taylor series.** The Taylor series is an analogue of the Taylor polynomial where we just go off to infinity. In symbols, if f is a function infinitely differentiable at 0, the Taylor series of f (also sometimes called the Maclaurin series or the Taylor-Maclaurin series) is:

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(0)x^k}{k!}$$

The *hope* is that for a nice enough function f and for x close enough to 0, the Taylor series evaluated at x actually converges to $f(x)$.

Note: Right now it is just a hope!

Note that truncating the Taylor series at any given n gives the Taylor polynomial P_n.

Let's first do some computational practice to see what the Taylor series look like, and then we shall proceed to understand what they mean.

1.3. **Taylor series for pedestrian functions.** For any function f for which we need to compute the Taylor series, we first compute the derivatives of f and get a sequence of functions. Next, we evaluate each of the derivatives at 0 to get a sequence of numbers. Next, we plug the elements of this sequence into the Taylor series.

First, let's consider a polynomial. The Taylor polynomials of a polynomial are really simple to understand: the n^{th} Taylor polynomial of a polynomial of degree m is the whole polynomial if $n \geq m$, and is the truncation of the polynomial to terms of degree $\leq n$ if $n < m$.

Let's now consider things more complicated than polynomials. For instance, consider the function exp.

The sequence of derivatives looks like:

$$\exp, \exp, \exp, \exp, \ldots$$

It is a *constant sequence of functions*. Evaluating at 0, we get the sequence:

$$1, 1, 1, 1, \ldots$$

And plugging into the Taylor series expression, we get:

$$\sum_{k=0}^{\infty} \frac{x^k}{k!}$$

which looks like:

$$1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \ldots$$

As we'll *see later*, it *turns out* that for the exp function, the Taylor series converges to $\exp(x)$ for any choice of x. In particular:

$$e = 1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \frac{1}{120} + \ldots$$

and:

$$2 = 1 + (\ln 2) + \frac{(\ln 2)^2}{2} + \frac{(\ln 2)^3}{6} + \ldots$$

and so on.

Similarly, we can determine the Taylor series for sin and cos. The sequence of derivatives for sin, starting from the 0^{th} derivative, is:

$$\sin, \cos, -\sin, -\cos, \sin, \cos, -\sin, -\cos, \ldots$$

The sequence of values at 0 is:

$$0, 1, 0, -1, 0, 1, 0, -1, \ldots$$

Because the sequence of functions has a period of 4, so does the sequence of values. The Taylor series thus turns out to be:

$$x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots$$

which is more compactly written as:

$$\sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!}$$

Similarly, the Taylor series for cos turns out to be:

$$1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots$$

which is more compactly written as:

$$\sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!}$$

It *turns out* that the series obtained above for sin and cos converge for all x to the function values. In particular, setting $x = \pi/2$ for the cos function:

$$0 = 1 - \frac{(\pi/2)^2}{2!} + \frac{(\pi/2)^4}{4!} - \dots$$

This can be useful for computing the value of 0 using the value of π.

1.4. **Some easy observations.** The big question that we have not yet tackled is whether the Taylor series converges, and if it does, whether it converges to the function. For each point that we make here, there is one justification using the formal definition (without assuming that the Taylor series converges to the original function) and another justification assuming that convergence actually occurs.

(1) For an even function, the Taylor series has nonzero coefficients only for even degree terms. For an odd function, the Taylor series has nonzero coefficients only for odd degree terms.

Justification in formal terms: Recall that the derivative of an even function is odd and the derivative of an odd function is even. Also, the value of an odd function at 0 is 0.

Thus, if f is even, the derivatives of f alternate between even and odd. Thus, all $f^{(k)}$ are odd for odd k, and thus $f^{(k)}(0) = 0$ for odd k. Hence, only the even degree terms of the Taylor series of f could potentially have nonzero coefficients.

Similarly, if f is odd, the derivatives of f alternate between even and odd. Thus, all $f^{(k)}$ are odd for even k, and thus $f^{(k)}(0) = 0$ for even k. Hence, only the odd degree terms of the Taylor series of f could potentially have nonzero coefficients.

Justification assuming convergence: If f is even, then $f(x) = f(-x)$. The same must be true of the Taylor series. Plugging in $-x$ in place of x, we see that the coefficients of odd degree terms must vanish. A similar argument works for f odd. This is similar to the argument that a polynomial is an even function if and only if it has only even degree terms.

(2) Suppose f is an infinitely differentiable function. The Taylor series for f' can be obtained through term wise differentiation on the Taylor series of f.

Justification in formal terms: The k^{th} term in the Taylor series expansion of f is:

$$f^{(k)}(0)x^k/k!$$

Differentiating this, we get:

$$f^{(k)}(0)kx^{k-1}/k!$$

Cancel the k and $k!$ to get:

$$f^{(k)}(0)x^{k-1}/(k-1)!$$

If $g = f'$, then $f^{(k)} = g^{(k-1)}$, so we obtain that the coefficient of x^{k-1} after term wise differentiation is:

$$g^{(k-1)}(0)/(k-1)!$$

which is the same as the coefficient in the Taylor series for $g^{(k-1)}$. Thus, the Taylor series for the derivative is obtained through term wise differentiation of the Taylor series for the function.

Justification assuming convergence: In this case, since the Taylor series converges to the function, differentiating the Taylor series is equivalent to differentiating the function. This isn't quite a complete and coherent argument, because we are differentiating an *infinite sum*, and we need to worry about additional convergence issues. But it gives the core reason.

(3) The Taylor series of the sum of two infinitely differentiable function is the term wise sum of their Taylor series. Analogous statements hold for the difference and the product.

Justification in formal terms: The statements for sums and differences follow from the fact that taking the k^{th} derivative is a linear operator, i.e., $(f+g)^{(k)}(0) = f^{(k)}(0) + g^{(k)}(0)$. The formulation for the product involves the use of the product rule.

Justification assuming convergence: Sums are sums, differences are differences, and products are products. Basically, if the series converges to the function, it makes sense that whatever operations we do to the functions, we need to do the same operations to the series.

(4) Suppose g is a polynomial function with zero constant term. The Taylor series for $f \circ g$ is the same as the series obtained by replacing x with $g(x)$ in the Taylor series of f.

For instance, if $g(x) = x^2$, then the Taylor series for $f(x^2)$ is the same as what we'd get by replacing x with x^2 in the Taylor series for f.

Thus, we get, for instance:

$$e^{-x^2} = 1 - \frac{x^2}{1!} + \frac{x^4}{2!} - \frac{x^6}{3!} + \dots$$

and:

$$\cos(ax) = 1 - \frac{a^2x^2}{2!} + \frac{a^4x^4}{4!} - \dots$$

Note that it is important that g have zero constant term, so that $g(0) = 0$.

Justification using formal definition: This arises from an irksome application of the chain rule and product rule for differentiation.

Justification assuming convergence: Assuming convergence, the Taylor series for $f \circ g$ should be the Taylor series for f, composed with g.

1.5. Taylor polynomials and Taylor series: more observations.

- The n^{th} Taylor polynomial of a function is of degree *at most* n, but the degree could be strictly less than n. That could happen if $f^{(n)}$ takes the value 0 at 0.
- The number of nonzero terms in the n^{th} Taylor polynomial is at most $n+1$, but it could be strictly less depending on which of the numbers $f(0)$, $f'(0)$, and so on till $f^{(n)}(0)$ are zero.
- We already computed the Taylor series for exp, sin, cos, and polynomials. The observations made above can now allow us to use these to compute the Taylor series for composites of these with polynomials, such as $\exp(-x^2)$, $\sin(x^2)$, as well as quotients of these by polynomials, for instance, $(\sin x)/x$.
- *Even and odd part*: Recall from long ago that for a continuous function f defined on all of $\mathbb{R}$, we can break f up in a unique manner as the sum of an even function f_e (called the *even part* of f) and an odd function f_o (called the *odd part* of f). In concrete terms, we define:

$$f_e(x) := \frac{f(x) + f(-x)}{2}$$

and:

$$f_o(x) := \frac{f(x) - f(-x)}{2}$$

The corresponding operation in terms of Taylor series is as follows: the Taylor series of the even part of a function is just the even degree terms (i.e., the even part) of the Taylor series, and the Taylor series of the odd part of a function is just the odd degree terms (i.e., the odd part) of the Taylor series.

Recall that cosh is the even part of exp and sinh is the odd part of exp. Thus, the Taylor series for cosh is the even psrt of the Taylor series for exp, and the Taylor series for sinh is the odd part of the Taylor series for exp. Explicitly, the Taylor series for $\cosh x$ is $\sum_{n=0}^{\infty} x^{2n}/(2n)!$ and the Taylor series for $\sinh x$ is $\sum_{n=0}^{\infty} x^{(2n+1)}/(2n+1)!$.

Aside: A linear operator from functions to series. [May not cover in class. Recommended for review time reading.]

This may (or may not!) clarify matters. Suppose we denote by $C^\infty(\mathbb{R})$ the set of all infinitely differentiable functions on $\mathbb{R}$. This set of functions is particularly nice: it is closed under pointwise addition, subtraction, and multiplication, as well as under scalar multiplication. It is also closed under differentiation. The closure under pointwise addition, subtraction, and scalar multiplication makes it a vector space over $\mathbb{R}$. Putting the multiplicative structure makes it an algebra over $\mathbb{R}$. Putting the differentiation in there as well, we get that $C^\infty(\mathbb{R})$ is an *algebra* over $\mathbb{R}$ equipped with a chosen differential operator.

Define $TS(\mathbb{R})$ as the set of all possible series of the form:

$$\sum_{k=0}^{\infty} a_k x^k$$

We can define on $TS(\mathbb{R})$ term wise addition and subtraction, term wise differentiation, and multiplication in a manner similar to polynomials. The addition, subtraction, and scalar multiplication makes $TS(\mathbb{R})$ a $\mathbb{R}$-vector space. Tacking on the multiplicative structure makes $TS(\mathbb{R})$ a $\mathbb{R}$-algebra. Tacking on the differentiation operator makes $TS(\mathbb{R})$ a $\mathbb{R}$-algebra with a chosen differential operator.

What we have done is given a mapping:

$$C^\infty(\mathbb{R}) \to TS(\mathbb{R})$$

which sends a given function f to its Taylor series.

The points above show that this mapping preserves sums, differences, products, and even derivatives. The fact that it preserves sums, differences, and scalar multiples makes it a *linear operator* between $C^\infty(\mathbb{R})$ and $TS(\mathbb{R})$. The fact that it preserves multiplication as well makes it an *algebra mapping* between $C^\infty(\mathbb{R})$ and $TS(\mathbb{R})$. The fact that it preserves the differentiation operator makes it an mapping of algebras with chosen differential operator.

In this sense, it is a very nice mapping. However, these points together do not mean that an element f in $C^\infty(\mathbb{R})$ goes to an element in $TS(\mathbb{R})$ that "means the same thing" as f. In particular, we don't know if the mapping described here is one-to-one. Can two different functions give rise to the same Taylor series? In linear algebra terms, what is the kernel of the mapping? It turns out, unfortunately, that the answer is that the function is *not one-to-one*, or has *nontrivial kernel*, which means that *different functions can have the same Taylor series*. Since a Taylor series can converge to only one function, it is *very much possible to have an infinitely differentiable function with a Taylor series that does not converge to it.* We deal with these issues in the coming sections.

2. Dealing with convergence

2.1. **Taylor polynomials.** We return for now to Taylor polynomials. Recall that the n^{th} Taylor polynomial P_n is a truncation of the Taylor series to a polynomial of degree n. We want to figure out, for a given function f with n^{th} Taylor polynomial P_n, how large is the gap:

$$|f(x) - P_n(x)|$$

Taylor's theorem answers this question, and we turn to it after a brief digression into approximation.

Although we will not prove the theorem in class, you are invited to look through the proof in the book (um, yeah, as if that ever worked!). You had a very similar question as one of your advanced problems earlier in this course.

2.2. **Linear fitting and curve fitting.** In this course, we defined the tangent line to a curve at a point as the *best linear approximation* to the curve at the point. This is because, if the curve has a particular slope (value of derivative of the function if the curve is the graph of the function), the tangent line with the same slope through that point comes closest to approximating the curve. If the curve is the graph of f and the point is $(0, f(0))$, we see that the equation of the tangent line is:

$$y = f(0) + f'(0)x$$

It is no coincidence that this is the first Taylor polynomial. Thus, the first Taylor polynomial is the *best linear approximation* to the graph of a function. This means that if we consider:

$$R_1(x) := f(x) - (f(0) + f'(0)x)$$

this difference is sub-linear, in the sense that it grows slower than linear. Another way of putting this is that R_1 has a zero at 0 of order *strictly greater than* 1. In other words, not only is $R_1(0)$ equal to zero, it changes very slowly around 0.

Taylor polynomials can be thought of as higher order generalizations of the tangent line. Basically, the n^{th} Taylor polynomial's task is to be the *best approximation by a polynomial of degree* n of the function f *near* 0. Remember that for x close to 0, x^n gets smaller and smaller. Thus, the larger we allow n to be, the finer the precision we can aim for.

Aside: Two meanings of best fit. The notion of *best fit* that we are using here is that of best fit to a function in terms of *local* behavior very close to a *particular point* (though that fit might also work far away, just by chance). It is a highly *local* notion of best fit.

Another notion of best fit, that is useful for capturing secular trends, is one of *global* best fit. For instance, we have a function with periodic derivative, and we know it can be written as a linear part plus a periodic part. We then select the linear part in such a manner that it most closely approximates the curve. More generally, there are techniques such as *Ordinary Least Squares* regression that allow one to find the best linear approximation to a bunch of scattered data.

Keep in mind that the notion of best fit for Taylor series is the local version, as opposed to the global version described in the previous paragraph.

2.3. **The remainder and Taylor's theorem.** For a function f that is (at least) $n+1$ times differentiable at 0, let P_n be the n^{th} Taylor polynomial about 0. We define the *remainder* as the following function:

$$R_n(x) := f(x) - P_n(x)$$

The remainder is thus what remains after we take out the n^{th} Taylor polynomial. Clearly, $R_n(0) = 0$.

Taylor's theorem states that, if f is $(n+1)$ times differentiable not just at 0 but in the interval from 0 to x, then:

$$R_n(x) = \frac{1}{n!}\int_0^x f^{(n+1)}(t)(x-t)^n\,dt$$

2.4. **The Lagrange formula and a max-estimate version.** The Lagrange formula is a corollary of Taylor's theorem, and it states that for any x, there exists c between 0 and x (so $c \in [0,x]$ if $x \geq 0$ and $c \in [x,0]$ if $x \leq 0$) such that:

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}x^{n+1}$$

This follows from Taylor's theorem and the mean value theorem.

The following max-estimate version is sometimes useful:

$$|R_n(x)| \leq \left(\max_{t \in J} |f^{(n+1)}(t)|\right)\frac{|x|^{n+1}}{(n+1)!}$$

where J is the interval joining 0 to x.

2.5. **Order interpretation.** The Lagrange formula implies that R_n has a zero of order at least $n+1$ at 0. In other words, it requires eyes of sensitively greater than n to see that R_n is not zero.

Recall that the order of a zero is the matching power of x at which the limit of the quotient breaks from zero to undefined. Also recall that the larger the order of zero, the more zeroish the zero.

2.6. **The convergence of Taylor series.** If we can show, for a given function f, that the remainder functions go to 0, i.e., that, for a given x_0:

$$\lim_{n \to \infty} R_n(x_0) = 0$$

then the Taylor series for f, evaluated at x_0, converges to x_0.

We can use this idea to show the statements I earlier asked you to take on faith: the Taylor series for exp, cos, and sin converge to the respective functions for all x. The chief reason for this is that exp, cos, and sin have the property that *the sequence of derivatives at any particular point is uniformly bounded.*

Let's illustrate with the example of sin. We have:

$$|R_n(x)| \leq \left(\max_{t \in J} |\sin^{(n+1)}(t)| \right) \frac{|x|^{n+1}}{(n+1)!}$$

where J is the interval between 0 and x.

Now, we know that any iterated derivative of sin is one of the four functions sin, cos, $-\sin$, and $-\cos$, and all of these have range $[-1, 1]$. Thus, the max-expression is at most 1. We thus get:

$$|R_n(x)| \leq \frac{|x|^{n+1}}{(n+1)!}$$

As $n \to \infty$, the right side approaches 0, because the denominator, being factorial, grows faster than the numerator, which is exponential in n. Thus, $|R_n(x)| \to 0$.

The same logic applies for cos. For exp, the logic is fairly similar:

$$|R_n(x)| \leq \left(\max_{t \in J} |\exp^{(n+1)}(t)| \right) \frac{|x|^{n+1}}{(n+1)!}$$

Any iterated derivative of exp is exp, so the max-expression is just $\max_{t \in J} |\exp(t)|$. This is either 1 or $\exp(x)$, depending on whether $x < 0$ or $x > 0$. In either case, it is *independent of* n. Thus, when we are taking the limit as $n \to \infty$, it pulls out of the limit and we effectively take the limit of $|x|^{n+1}/(n+1)!$, which still tends to 0.

In the next lecture, we will introduce some terminology and concepts that will help us consider the collection of functions for which the power series does converge to the original function. The upshot will be that this collection is itself closed under addition, subtraction, multiplication, scalar multiplication, and differentiation.

2.7. **The problems that could occur in general.** In general (i.e., for functions other than exp, cos, and sin), two problems could happen:

(1) The Taylor series does not converge: Sometimes, it does not converge for any x. Sometimes it converges for some x and not others.
(2) The Taylor series converges but the sum of the Taylor series is not the original function: This is rare, but it does happen for some artificially concocted functions. It does not, however, happen for most of the simplest functions we will be dealing with.

We will tackle both these problems later.

3. Taylor series in $x - a$

So far, we have been interested in the local behavior near zero, and hence, we have been handling Taylor series about 0. The more general version involves Taylor series about some arbitrary point $a \in \mathbb{R}$, such that the function is defined and infinitely differentiable about a. The series is:

$$\sum_{k=0}^{\infty} f^{(k)}(a) \frac{(x-a)^k}{k!}$$

The evaluation point now becomes 0 instead of a and we replace x by $x - a$. We thus get a series where each term is a multiple of a power of $x - a$.

The truncation of these to degree n polynomials are called the Taylor polynomials.

There is nothing really new here – we can achieve everything by considering a new function that arises as the old function shifted by a. So, Taylor's theorem and the Lagrange formula all have analogues in this context, which you can read from the book (12.7.1 – 12.7.3). It might, however, be an instructive exercise for you to first try to guess at the general statement by looking at the statement in the special case $a = 0$, and then confirm your guess against the book.