SUMMATION TECHNIQUES

MATH 153, SECTION 55 (VIPUL NAIK)

Corresponding material in the book: Scattered around, but the most cutting-edge parts are in Sections 12.8 and 12.9.

What students should definitely get: The myriad techniques for summing up series of various kinds, and how these interact.

Executive summary

Here are useful guidelines on summing up functions (i.e., finding the actual sums, not just determining convergence):

(1) For finite sums involving polynomials of small degree, use linearity and the formulas for summations of 1, k, k^2, k^3.
(2) For reciprocals of quadratic functions, use partial fractions and then look for telescoping when the quadratic can be factorized. If the quadratic cannot be factorized but is a perfect square, try to use $\zeta(2) = \pi^2/6$. If the quadratic has negative discriminant, there is no closed form expression.
(3) In general, look for telescoping wherever you go. This includes rational functions, logarithms (e.g., $\ln((k+1)/k)$).
(4) Sometimes, for higher degree rational functions, you can combine telescoping with known information about zeta functions.
(5) See if the summation is a geometric series in disguise, or combines two or more geometric series and some possibly anomalous terms.
(6) Sometimes, the summation is related to a geometric series via integration or differentiation. For instance $\sum kx^k$ is related to $\sum x^k$ via differentiation. Use the differentiation and integration theorems to use these to get closed forms.
(7) In some cases, the summation is a known series such as that for the exponential, sine, cosine, arc tangent or logarithm, with some modifications: it might involve a sum or difference of two such series, it might be arrived at by composing such a series with mx^n, it might be arrived by multiplying such a series with mx^n, it might be arrived at by integrating or differentiating such a series.
(8) To identify these possibilities better, here are some heuristics: factorials in denominator suggests exponentials or sine/cosine, and the nature of sign alternation helps decide which. Ordinary k in the denominator suggests logarithm or arc tangent, and the nature of sign alternation helps decide which. [Exponential and logarithm have a sign periodicity of at most 2, while sine, cosine and arc tangent have a sign periodicity of 4].

1. Summation techniques with polynomial and rational function coefficients

We break the discussion into two parts. First, we discuss summation techniques for series where the terms of the series are given by rational functions. Next, we discuss summation techniques for power series where the coefficients are given by rational functions.

1.1. **The case of summing up polynomials.**

For summing up polynomials of degree up to 3, we can use linearity of summation along with what we know about summation formulas for 1, k, k^2, and k^3:

$\sum_{k=1}^n 1 = n$, $\sum_{k=1}^n k = n(n+1)/2$, $\sum_{k=1}^n k^2 = n(n+1)(2n+1)/6$, $\sum_{k=1}^n k^3 = n^2(n+1)^2/4$.

Note that the infinite summation for any polynomial gives $+\infty$ or $-\infty$ depending upon the sign of the leading coefficient.

1.2. **Summing up rational functions.** We first consider the case of a quadratic denominator that can be factored. If the summation is of the form:

$$\sum_{k=1}^{n} \frac{1}{(k-\alpha)(k-\beta)}$$

Then the sum can be telescoped using the techniques of partial fractions if α and β differ by a nonzero integer. In particular, we can use that telescoping to determine the infinite sum:

$$\sum_{k=1}^{\infty} \frac{1}{(k-\alpha)(k-\beta)}$$

This, however, is not good enough to tackle all summations with quadratic denominators. For summations of the form:

$$\sum_{k=1}^{\infty} \frac{1}{k^2}$$

use the fact that this sum is $\zeta(2) = \pi^2/6$.

A related summation is:

$$\sum_{k=1}^{\infty} \frac{1}{(2k-1)^2}$$

We can work out that this is the sum of reciprocals of all squares minus the sum of reciprocals of the squares of even numbers, giving $\pi^2/6 - \pi^2/24 = \pi^2/8$.

Most other quadratic summations cannot be resolved using the techniques we have seen so far.

For any summation that we cannot directly resolve, we can use the concrete version of the integral test to estimate the sum to infinity or within a finite interval. *Note that the concrete version requires the function to be non-increasing on the interval to which it is applied*, hence it may not be applicable starting 0 or 1, and we may need to move to a later point to start it.

1.3. **Summing up reciprocals of linears: two special cases.** We have two special summations that arise from the power series stuff and are not obvious otherwise:

$$\ln 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k}$$

and:

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots = \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1}$$

These two special cases can be used to do some alternating infinite summations.

1.4. **Power series summations.** For power series with rational function coefficients, the radius of convergence is always 1. Convergence at the endpoints is determined either by comparison with a p-series (or the degree difference heuristic) or using the alternating series theorem.

there are the following formulas of note:

$$\sum_{k=0}^{\infty} x^k = 1/(1-x)$$

with radius of convergence 1, does not converge at either endpoint.

Also:

$$\sum_{k=1}^{\infty} x^k/k = -\ln(1-x)$$

with radius of convergence 1, converges at the endpoint -1 to $-\ln 2$ but not at the endpoint 1.

We can deduce from these a number of other formulas, such as expansions for $\ln(1+x)$, $\ln((1+x)/(1-x))$, and more. In particular, $\ln((1+x)/(1-x))$ has the power series:

$$\ln((1+x)/(1-x)) = 2(x + \frac{x^3}{3} + \frac{x^5}{5} + \dots)$$

The interval of convergence is $(-1, 1)$.

We also have an expansion for arctan:

$$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \dots$$

with radius of convergence 1 and interval of convergence $[-1, 1]$.

We can combine these facts to see that:

$$\frac{\arctan x}{2} + \frac{\ln((1+x)/(1-x))}{4} = x + \frac{x^5}{5} + \frac{x^9}{9} + \dots$$

1.5. **Applying these to summations with quadratic denominators.** We can perform the summation:

$$\sum_{k=1}^{\infty} \frac{1}{k(k-1/2)}$$

Let's perform this summation as an illustration. First, we rewrite this as:

$$\sum_{k=1}^{\infty} \frac{4}{(2k)(2k-1)}$$

Now we use partial fractions to rewrite this as:

$$4\sum_{k=1}^{\infty} \left[\frac{1}{2k-1} - \frac{1}{2k}\right]$$

Rewrite this and notice that we get the summation for $\ln 2$ in there, so we get $4\ln 2$. (This needs a little more justification, since the summation for $\ln 2$ is not absolutely convergent, so it is critical that our regrouping is not affecting the sum).

1.6. **Index off by a constant.** We know that:

$$\sum_{k=1}^{\infty} \frac{x^k}{k} = -\ln(1-x), |x| < 1$$

We can use this to calculate summations such as:

$$\sum_{k=1}^{\infty} \frac{x^k}{k+1}$$

For this kind of summation, the critical thing to note is that the denominator cannot be manipulated easily, so we manipulate the numerator instead. Specifically, for $x \neq 0$, we rewrite this as:

$$\sum_{k=1}^{\infty} \frac{1}{x} \frac{x^{k+1}}{k+1}$$

The $1/x$ can be pulled out. Putting $l = k+1$, we get:

$$\frac{1}{x}\sum_{l=2}^{\infty} \frac{x^l}{l}$$

Note that l now starts at 2. The summation is now almost the summation for $-\ln(1-x)$, but not quite, because it is missing the first term. Thus, it is $-\ln(1-x) - x$, and we get:

$$\frac{-\ln(1-x)-x}{x}$$

Note that this expression is not valid for $x = 0$. However, using the LH rule, we can verify that its limit at $x = 0$ is 0, which is the same as the value of the summation when we plug in $x = 0$.

We can use these ideas to calculate things like:

$$\sum_{k=1}^{\infty} \frac{x^k}{k(k+1)}$$

For more details on this worked examples, see the quiz on series and convergence.

1.7. **Index off by a multiple.** Consider the summation:

$$\sum_{k=1}^{\infty} \frac{x^{3k}}{k}$$

Here, even though the denominators cover all the natural numbers, the exponent in the numerator is growing by steps of 3. In this case, we put $u = x^3$, and get:

$$\sum_{k=1}^{\infty} \frac{u^k}{k}$$

This simplifies to $-\ln(1-u) = -\ln(1-x^3)$.

1.8. **A more challenging problem.** Consider the summation:

$$\sum_{k=2}^{\infty} \frac{1}{k^4 - k^2}$$

First, we factorize the denominator:

$$\sum_{k=2}^{\infty} \frac{1}{k^2(k^2-1)}$$

As a first pass, we break partly into partial fractions:

$$\sum_{k=2}^{\infty} \left[\frac{1}{k^2-1} - \frac{1}{k^2}\right]$$

The full break-up is:

$$\sum_{k=2}^{\infty} \left[\frac{1/2}{k-1} - \frac{1/2}{k+1} - \frac{1}{k^2}\right]$$

Now, since $\sum 1/k^2$ is absolutely convergent it can be split off. Since the summation starts at 2, this gives $\pi^2/6 - 1$. This leaves:

$$\frac{1}{2}\sum_{k=2}^{\infty} \left[\frac{1}{k-1} - \frac{1}{k+1}\right]$$

After telescoping and taking limits, we get $(1/2)(1+1/2) = 3/4$. The overall answer is thus $3/4 - (\pi^2/6 - 1) = 7/4 - \pi^2/6$.

1.9. **Summations with polynomial coefficients.** We know that:

$$\sum_{k=0}^{\infty} x^k = \frac{1}{1-x}$$

Differentiating, we get:

$$\sum_{k=1}^{\infty} kx^{k-1} = \frac{1}{(1-x)^2}$$

Relabeling gives:

$$\sum_{k=0}^{\infty} (k+1)x^k = \frac{1}{(1-x)^2}$$

Now subtracting the original gives:

$$\sum_{k=0}^{\infty} kx^k = \frac{x}{(1-x)^2}$$

We can do another differentiation and subtraction to get formulas for $\sum k^2 x^k$. All these formulas are valid for $|x| < 1$. Once we have these basic formulas, we can again use linearity of summation.

2. Summations involving factorials

We have:

$$\begin{aligned}
\exp(x) &= \sum_{k=0}^{\infty} \frac{x^k}{k!} \\
\exp(-x) &= \sum_{k=0}^{\infty} \frac{(-1)^k x^k}{k!} \\
\cosh x &= \sum_{k=0}^{\infty} \frac{x^{2k}}{(2k)!} \\
\sinh x &= \sum_{k=0}^{\infty} \frac{x^{2k+1}}{(2k+1)!} \\
\cos x &= \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!} \\
\sin x &= \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!} \\
(\cosh x + \cos x)/2 &= \sum_{k=0}^{\infty} \frac{x^{4k}}{(4k)!} \\
(\cosh x - \cos x)/2 &= \sum_{k=0}^{\infty} \frac{x^{4k+2}}{(4k+2)!} \\
(\sinh x + \sin x)/2 &= \sum_{k=0}^{\infty} \frac{x^{4k+1}}{(4k+1)!} \\
(\sinh x - \sin x)/2 &= \sum_{k=0}^{\infty} \frac{x^{4k+3}}{(4k+3)!}
\end{aligned}$$

The common feature in all these summations is that the exponent on x equals the quantity whose factorial appears in the denominator.

If the exponent is off by a constant, it can be adjusted. For instance:

$$\sum_{k=0}^{\infty} \frac{x^k}{(k+1)!}$$

can be rewritten as:

$$\frac{1}{x} \sum_{l=1}^{\infty} \frac{x^l}{l!}$$

This becomes $(e^x - 1)/x$. The -1 there arises because we are missing the 0^{th} term.

If the exponent is off by a multiple, then we need to substitute to a power or fraction:

$$\sum_{k=0}^{\infty} \frac{x^{2k}}{k!}$$

is e^{x^2}. Similarly:

$$\sum_{k=0}^{\infty} \frac{x^k}{(2k)!}$$

is $\cosh \sqrt{x}$ when $x > 0$, and $\cos \sqrt{-x}$ when $x < 0$ *Note: This sentence was corrected.* For the case $x = 0$, both definitions apply.

Note that the nature of growth of the factorials in the denominator as well as the nature of sign alternation tell us about the kind of thing the summation goes to. In particular:

- If the inputs to the factorials in the denominator jump by one and there is no sign alternation, we have exp. Instead of $\exp(x)$, we may have things like $\exp(x^m)$ or $((\exp(x) - 1)/x$.
- If the inputs to the factorials in the denominator jump by one and there is sign alternation, we have something like $\exp(-x)$. We may have $\exp(-x^2)$.
- If the inputs to the factorials in the denominator jump by two and there is no sign alternation, then we have something based on cosh if the inputs are even and something based on sinh if the inputs are odd. Examples are $\cosh(x^2)$, $(1 - \cosh(x))/x^2$, $\sinh(x^4)$, $\cosh(\sqrt{|x|})$, etc.
- If the inputs to the factorials in the denominator jump by two and there is sign alternation, then we have something based on cos if the inputs are even and something based on sin if the inputs are odd. Examples are $\sin(x^2)$, $(\sin x)/x$, $(1 - \cos x)/x^2$, etc.
- If the inputs jump by 4 and there is no sign alternation, then we have something based on one of these: $(\cosh + \cos)/2$ (if all inputs are multiples of 4), $(\cosh - \cos)/2$ (if all inputs have a remainder of 2 mod 4), $(\sinh + \sin)/2$ (if all inputs have a remainder of 1 mod 4) and $(\sinh - \sin)/2$ (if all inputs have a remainder of 3 mod 4).

3. Summation where the coefficients have a geometric component

If the coefficients in a summation have a part which is just a^k for a constant a, we can do the substitution $u = ax$ to absorb this constant into x, and hence eliminate the geometric component of the summation.

For instance:

$$\sum_{k=1}^{\infty} \frac{x^k}{2^k k}$$

Put $u = x/2$ and we get $\sum_{k=1}^{\infty} u^k/k$, for which we have a formula. Note that the radius of convergence for x is not the same as that for u, because $x = 2u$. Rather, the radius of convergence for u is 1 whereas the radius of convergence for x is 2.

4. Bounding summations, estimating growth

It is extremely hard to do summations in general, apart from the few cases mentioned here. However, ideas such as concrete versions of the basic comparison test can be used to bound an existing summation in terms of the others.

In the discussion here, we will assume that we are restricting to inputs $x > 0$, and all the power series coefficients are nonnegative. Further, our interest is in how the summation grows as $x \to \infty$.

In general, the larger the terms being added, the larger the summation gets. Recall that we had the following hierarchy of functions we used for determining radius of convergence:

(1) Double exponential and other such monstrosity.
(2) Exponential in $x^r, r > 1$.
(3) Factorial, or Γ function. Roughly, exponential in $x \ln x$.
(4) Exponential or geometric.
(5) Exponential in $x^r, r < 1$.
(6) Exponential in $(\ln x)^r, r > 1$.
(7) Polynomial, or about $x^r, r > 0$.
(8) Polynomial in the logarithm.

4.1. The case of very big denominators.

In particular, we know that if the denominator is of types (1), (2), and (3) within this hierarchy (with the numerator being a constant or of type (4) or lower), then the series converges (and converges absolutely) everywhere on $\mathbb{R}$. However, the only cases for which we have actual formulas are the cases where the denominators are growing as factorials. That's what we just saw in the section just concluded.

When the denominators are of types (1) and (2), then the terms being added (being reciprocals of their denominators) are *smaller* than the terms of an exponential series. Hence, a power series with denominators of types (1) or (2) grows *sub-exponentially*. However, assuming that there are infinitely many terms, it must grow *super-polynomially*. This is roughly all we can say offhand.

Thus, for instance, the function f defined as:

$$f(x) := \sum_{k=0}^{\infty} \frac{x^k}{2^{k^2}}$$

grows sub-exponentially but super-polynomially in x as $x \to \infty$, because the denominator, being of type (2), is growing at a rate faster than anything factorial or factorial-like.

The same is true for the function g defined as:

$$g(x) := \sum_{k=0}^{\infty} \frac{x^k}{2^{3^k}}$$

grows sub-exponentially, because the denominator is of type (1).

4.2. The case of denominators that are sort of like type (3).

In some cases, the denominator has a growth rate comparable to factorials, but it does not precisely match any of the formulas we have seen. For instance:

$$\sum_{x=0}^{\infty} \frac{x^k}{(k!)^2}$$

Here, the denominator is the square of $k!$, which grows faster than $k!$. But it still grows slower than $(2k)!$. We can thus bound the summation between $\cosh(\sqrt{x})$ and e^x.

Similarly, consider the summation:

$$\sum_{k=0}^{\infty} \frac{x^k}{(3k)!}$$

Here, if we put $u = x^{1/3}$, we get:

$$\sum_{k=0}^{\infty} \frac{u^{3k}}{(3k)!}$$

This is basically a subseries of the exponential series, picking out every third term. Hence, it is bounded from above by $\exp(u)$, which is $\exp(x^{1/3})$. It can also be bounded from below in terms of $\exp(x^{1/3})$, but that bounding procedure is trickier and we skip over it.

4.3. **The case of denominators that are type (4) or slower.** In this case, the radius of convergence is finite, so it does not make sense to talk of the behavior at or near infinity.