SUMMATION NOTATION, SERIES, AND CONVERGENCE BASICS

MATH 153, SECTION 59 (VIPUL NAIK)

Corresponding material in the book: Section 12.1, 12.2, 12.3.

What students should definitely get: The summation notation and how it works, series, concepts of convergence. The use of telescoping and forward difference operator ideas to sum up series. The use of the integral test and other tests to determine whether a series converges and obtain numerical estimates. Convergence rules for rational functions.

What students should hopefully get: How the summation notation is similar to the integral notation, how the parallels can be worked out better.

Executive summary

Words ...

(1) *Not for discussion*: The $\sum$ summation notation is used to compactly express a sum of many (finitely or infinitely many) terms. The terms of the summation are called *summands* and the variable that changes value across summands is termed the *variable of summation* or *index of summation*, and is a *dummy variable*. Some variants include: (i) writing the start of summation at the bottom and the end of summation at the top, (ii) writing the set constraint at the bottom, (iii) doing (i) or (ii) but omitting the index of summation.

(2) *Not for discussion*: The infinite sum $\sum_{k=1}^{\infty} f(k)$ is defined as the limit $\lim_{n\to\infty} \sum_{k=1}^{n} f(k)$. The sums $\sum_{k=1}^{n} f(k)$ are termed the *partial sums*. We use the term *series* for a sequence to be summed up. The sum of a series is the limit of the sequence of partial sums. The summands are called the *terms* of the series.

(3) For a series of nonnegative terms, the sum is independent of the ordering of the terms. It can also be determined by grouping together the terms in any manner whatsoever. Thus, sums of nonnegative terms are commutative and associative in a strong sense.

(4) Summation is linear: it is additive and scalar multiples can be pulled out. In other words, $\sum(f(k)+g(k)) = \sum f(k) + \sum g(k)$. On the other hand, summation is *not* multiplicative. In other words, $\sum f(k)g(k)$ is not the same thing as $(\sum f(k))(\sum g(k))$.

(5) If $g = \Delta f$ where Δ is the forward difference operator, then $\sum_{k=a}^{b} g(k) = f(b+1) - f(a)$. This has a more general version called telescoping. Telescoping can be thought of as the discrete analogue of the fundamental theorem of calculus.

(6) There are four kinds of things we can do concretely for a series of nonnegative terms: (i) show that the series diverges, (ii) show that the series converges, and find its sum, (iii) show that the series converges, and find bounds on its sum, without finding an explicit summation-free expression for the sum, (iv) show that the series converges, without any explicit bounds on its sum.

(7) A series of nonnegative terms converges to the least upper bound of its sequence of partial sums. Note that the sequence of partial sums is a non-decreasing sequence precisely because the terms of the series (i.e., the summands) are nonnegative.

(8) If a series of (possibly mixed sign) terms converges, the magnitudes of the terms must go to 0. The contrapositive is that if the terms of a series do not go to 0, the series does not converge. This establishes a *necessary but not sufficient condition* for the convergence of a series.

(9) Left shifts and/or changing finitely many terms does not change the convergence of a series though it may change the value of the sum of the series. This result holds for a series with mixed sign terms.

(10) A geometric series is a series where the quotient of successive terms is constant. The constant (successor term over current term) is termed the *common ratio*. A geometric series of possibly mixed

sign terms converges if and only if the common ratio has absolute value strictly less than 1. If the first term is a and the common ratio is r, the geometric series converges to $a/(1-r)$.

(11) The sum of a *finite* segment of a geometric series with n terms, first term a, and common ratio r, is $a(1-r^n)/(1-r)$ if $r \neq 1$, and na if $r = 1$.

(12) The integral test gives both a computational estimate for the sum of a series and a conditional test for whether the series converges. In particular, it states that for a (eventually) nonnegative, (eventually) continuous, (eventually) decreasing function, the integral is finite if and only if the sum is. (See the class notes for the details on the computational estimate; the book concentrates on the conditionality aspect).

(13) The basic comparison test states that if $0 \le a_k \le b_k$ for all sufficiently large k, and $\sum b_k$ converges, so does $\sum a_k$. Similarly, if $\sum a_k$ diverges, so does $\sum b_k$.

(14) The limit comparison test states that if $\lim_{k\to\infty} a_k/b_k$ is finite and positive, then both series have the same convergence/divergence behavior.

(15) For $p > 0$, consider the p-series $\sum_{k=1}^{\infty} k^{-p}$. This series diverges for $p \le 1$ and converges for $p > 1$. We define the zeta function $\zeta(p)$ as the sum of the series. We note that $\max\{1, 1/(p-1)\} \le \zeta(p) \le p/(p-1)$ for all p, by the integral test. [Review how this is derived]. ζ is a continuous decreasing function on $(1, \infty)$ with $\lim_{p\to 1^+} \zeta(p) = \infty$ and $\lim_{p\to\infty} \zeta(p) = 1$. Also, $\zeta(2) = \pi^2/6$ and $\zeta(4) = \pi^4/90$. [Note: We have *not* seen how to derive these values.]

Actions ...

(1) Telescoping is a powerful tool. It allows us to use partial fractions to sum up various kinds of sums with quadratic denominatorics. In particular, if $g(k) = f(k) - f(k+m)$, with $f(k) \to 0$ as $n \to \infty$, then $\sum g(k) = f(1) + f(2) + \cdots + f(m)$.

(2) We can also use the known formulas for summing up $\sum 1$, $\sum k$, $\sum k^2$ and $\sum k^3$ along with linearity to calculate summations where the summands are polynomials of degree at most 3 in k.

(3) We can sum up an *eventually* geometric series by summing up the eventual geometric part of the series and separately handling the first few anomalous terms. *It often happens in real world series that the series is eventually geometric but the first few terms are anomalous. This is due to boundary effects/startup issues.* [For instance, the bouncing ball distance traveled problem.]

(4) For a series with alternating common ratios, we can sum up by splitting into two geometric series. (see example from class notes).

(5) The geometric series can be interpreted as an expansion for $1/(1-x)$:

$$\frac{1}{1-x} = 1 + x + x^2 + \ldots, |x| < 1$$

As we see later, this is the Taylor series for the function $1/(1-x)$.

(6) As a corollary of the convergence results on p-series and the basic comparison test, we have the following rule: for a rational function f, the series $\sum f(k)$ converges if the degree of the denominator exceeds the degree of the numerator by at least 2, and diverges otherwise.

(7) More generally, for series involving polynomial-like things, if the degree of the denominator (possibly fractional) exceeds the degree of the numerator by something strictly greater than 1, the series converges, otherwise it diverges.

(8) Things get a little trickier when $\ln k$ appears in the terms. ln can be thought of as being a polynomial in k of degree 0^+ – slightly greater than 0 but less than any positive number. If, in the given setup, the degree of the denominator minus the degree of the numerator is strictly less than 1, 1^- or 1, the series diverges. If it is strictly greater than 1, the series converges. If it is 1^+, then the situation is indeterminate, and we may need to use the integral test. For instance, $\sum(\ln k)/k$ clearly diverges, because the difference is 1^-. On the other hand $\sum 1/(k\ln(k+1))$ and $1/(k[(\ln k)^2+1])$ are ambigious, because we get 1^+ in both cases. In fact, the series diverges in the former case and converges in the latter case, as we can see using the integral test.

(9) sin and cos are bounded between -1 and 1, and this, along with the basic comparison test, can often be used to ascertain behavior about these functions. For instance, consider $\sum(2 + \sin k)/k^2$.

1. The summation notation: review

Suppose we want to write:

$$1^2 + 2^2 + 3^2 + \cdots + n^2$$

The "..." (ellipses) in between are somewhat ambiguous. Since we're good mind readers, we know what is meant. However, it would be better to have a notation that allows us to compactify this while removing the ambiguity. More generally, for a function f defined on $\{1, 2, 3, \ldots, n\}$, we want a shorthand notation for:

$$f(1) + f(2) + \cdots + f(n)$$

The shorthand notation is:

$$\sum_{k=1}^{n} f(k)$$

Here, k is a *dummy variable* called the *index of summation*. The expression $k = 1$ written under the $\sum$ symbol tells us where we start k off. The n on top of the $\sum$ symbol tells us the *last* value of k that we use. The default increment is 1.

Similarly, the summation:

$$\sum_{k=5}^{8} 2^k$$

is shorthand for the summation:

$$2^5 + 2^6 + 2^7 + 2^8$$

The $k =$ is sometimes eliminated, when there is clearly only one dummy variable and there is no scope for confusion. So, we can write the above summation as:

$$\sum_{5}^{8} 2^k$$

We can also start the summation from 0; for instance:

$$\sum_{k=0}^{6} k^3$$

Aside: For loops. For those of you who have dealt with for loops in the context of computer programming, the summation notation is a lot like a for loop. The expression below the $\sum$ sign is the initial condition for the dummy variable in the for loop, the default increment is $+1$, and the expression above the $\sum$ sign is the value at the *last iteration* – once we cross this value, we exit the summation.

1.1. Slightly different notation for summation.

A slightly different summation notation is where we describe the entire set of summation below the $\sum$ sign. *Unless otherwise specified or clear from context, the index of summation takes integer values only.* For instance:

$$\sum_{1 \le k \le 5} (2^k - k + 1)$$

means that we sum up the expression $2^k - k + 1$ for k in the set $\{1, 2, 3, 4, 5\}$. This is thus:

$$(2^1 - 1 + 1) + (2^2 - 2 + 1) + (2^3 - 3 + 1) + (2^4 - 4 + 1) + (2^5 - 5 + 1)$$

We can also specify the set of values of k; for instance:

$$\sum_{k \in \{1,4,6\}} k^3$$

This is shorthand for $1^3 + 4^3 + 6^3$.

We can shorten the above even further, by writing it as:

$$\sum_{\{1,4,6\}} k^3$$

1.2. **The parallel with integration notation.** The summation notation is similar to the integration notation. Consider for a function f:

$$\sum_{k=a}^{b} f(k)$$

versus the integral:

$$\int_a^b f(x)\,dx$$

In the former, we literally add up the values of $f(k)$ for $k = a, a+1, a+2, \ldots, b$. In the latter, we are integrating a continuous function over the closed interval $[a, b]$. The former is a discrete summation of finitely many values. The latter is a continuous summation. Integration is continuous summation and summation is discrete integration.

There are, however, a few crucial differences between summation and integration. Most importantly, integration is insensitive to a change in the function value at one point, because we are adding up infinitely many values. Summation, on the other hand, is sensitive to each value.

1.3. **Good notation tip.** An integral sign $\int$ is like an opening parentheses, and its corresponding closing parentheses is a dx (or d-whatever dummy variable we have). The part between these is the integrand.

A summation $\sum$ is also an opening parenthesis, but it has no corresponding closing parenthesis. In other words, there is no standard convention to denote where the expression being summed (called the *summand*) ends. It is thus good practice to put the entire summand in parentheses if there is some additional content that appears after the summand ends. For instance:

$$\sum_{k=1}^{n} k^2 + n^2$$

could mean:

$$\left[\sum_{k=1}^{n} k^2\right] + n^2$$

but it could also mean:

$$\sum_{k=1}^{n} (k^2 + n^2)$$

1.4. **The forward difference operator and summation.** Recall that for a function f on $\mathbb{N}$, the forward difference operator Δ gives the function Δf defined by $(\Delta f)(n) = f(n+1) - f(n)$. Given $\Delta f = g$, what is f? It turns out that an analogue of the fundamental theorem of calculus holds:

$$f(n) = f(1) + \sum_{k=1}^{n-1} g(k)$$

So f is the summation of its difference operator. Just like the integral of its derivative.

2. Infinite sums

An infinite sum is defined as a limit of the corresponding finite sums. Thus, the infinite sum:

$$\sum_{k=1}^{\infty} f(k)$$

is defined as:

$$\lim_{n\to\infty} \sum_{k=1}^{n} f(k)$$

if the limit exists. In other words, the infinite sum here is the limit of the finite sums. These finite sums are sometimes called the corresponding *partial sums*.

If the infinite sum exists, then we say tat the series *converges*.

For a sequence $a_1, a_2, \dots$, we write the following *series* as:

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \dots + a_n + \dots$$

The n^{th} *partial sum* of the series is the sum of the first n terms of the series. The *sum* of the series is the limit of the partial sums. In fact, studying the convergence of a series is equivalent to studying the convergence of the *sequence of its partial sums*. Explicitly, we define the sequence (s_n) of partial sums, where:

$$s_n = \sum_{k=1}^{n} a_k$$

2.1. Is the sum just the sum?

Is the sum of a series just our intuitive notion for the total value of all the elements of the series? Let us poke our intuitions to figure out what we intuitively think of as the total. One thing we certainly expect about the total is that it is commutative and associative: it is independent of the ordering of terms and the groupings we use for the terms. This means that if we just permute the terms, the sum should be invariant if it means what we think it means.

It is possible to have series that do not satisfy this property. However, if all the terms of the series are nonnegative, then the sum is invariant under rearrangements. This result is the *rearrangement theorem*, that we shall talk about a little later.

3. Some examples of finite and infinite sums and the methods used

3.1. Telescoping.

Suppose we want to find:

$$\sum_{k=a}^{b} g(k)$$

Additive telescoping involves finding a function f such that $\Delta f = g$. In other words, we find a function f such that:

$$g(k) = f(k+1) - f(k)$$

We can thus write the summation as:

$$[f(a+1) - f(a)] + [f(a+2) - f(a+1)] + \dots + [f(b+1) - f(b)]$$

This simplifies to:

$$f(b+1) - f(a)$$

The explanation is that, apart from the $-f(a)$ in the first term and the $f(b+1)$ at the end, everything cancels out.

This is *just like* the fundamental theorem of calculus. Here, f is the discrete analogue of an *antiderivative* for g, and to add the g-values over an interval, we evaluate f at the endpoints and take the difference. However, the *discrete nature of the situation* makes things slightly different: instead of $f(b) - f(a)$, we get $f(b+1) - f(a)$.

For instance, consider $g(k) = \frac{1}{k(k+1)}$.

Then, we have:

$$g(k) = \frac{1}{k} - \frac{1}{k+1}$$

Here, $f(k) = -1/k$, and we get that the summation from a to b is $(1/a) - (1/(b+1))$.

Similarly, consider:

$$g(k) = 2k + 1$$

We note that $g(k) = (k+1)^2 - k^2$, so we get $f(k) = k^2$, and we get:

$$(b+1)^2 - a^2$$

Thus, to carry out summations in general, we need to find these discrete antiderivatives. This is generally a hard task.

The term *telescoping* is sometimes used in a looser sense, where we try to find f such that $g(k) = f(k) - f(k+m)$ for some m. For instance, consider:

$$\sum_{k=1}^{10} \frac{1}{k(k+2)}$$

Using partial fractions, we can rewrite this as:

$$\frac{1}{2} \sum_{k=1}^{10} \frac{1}{k} - \frac{1}{k+2}$$

Let's write the first few terms to see how the telescoping occurs:

$$\frac{1}{2}[(1 - (1/3)) + ((1/2) - (1/4)) + ((1/3) - (1/5)) + \cdots + ((1/9) - (1/11)) + ((1/10) - (1/12))]$$

Notice what terms cancel out: everything except the 1 and the $1/2$ in the beinning and the $-1/11$ and $1/12$ at the end, so we get:

$$\frac{1}{2}\left[1 + \frac{1}{2} - \frac{1}{11} - \frac{1}{12}\right]$$

This simplifies to $175/264$ (?).

In general, if $g(k) = f(k) - f(k+m)$, we are left with:

$$\sum_{k=1}^{m} [f(a+k-1) - f(b+k)]$$

This is still a summation, but if m is considerably smaller than $b - a$, then it is a summation over a much smaller collection.

3.2. **Linearity.** The linearity of summations allows us to split a summation of a sum of two functions as the sum of their respective summations. It also allows us to pull out constants. In symbols:

$$\begin{aligned}\sum_{k=a}^{b}[f(k)+g(k)] &= \sum_{k=a}^{b} f(k) + \sum_{k=a}^{b} g(k)\\ \sum_{k=a}^{b}[\lambda f(k)] &= \lambda \sum_{k=a}^{b} f(k)\end{aligned}$$

Thus, if we know the discrete antiderivatives (i.e., summations) of all functions n^r, we can calculate summations for all polynomials.

Unfortunately, these discrete antiderivatives are not as pretty as their continuous counterparts. There is no easy general formula. But we can get started:

$$\begin{aligned}\sum_{k=1}^{n} 1 &= n\\ \sum_{k=1}^{n} k &= \frac{n(n+1)}{2}\\ \sum_{k=1}^{n} k^2 &= \frac{n(n+1)(2n+1)}{6}\\ \sum_{k=1}^{n} k^3 &= \frac{n^2(n+1)^2}{4}\end{aligned}$$

We can thus do summations for polynomials of degree up to three using these formulas. For instance:

$$\sum_{k=1}^{n}(k^2+2k+7) = \sum_{k=1}^{n} k^2 + 2\sum_{k=1}^{n} k + 7\sum_{k=1}^{n} 1 = \frac{n(n+1)(2n+1)}{6} + n(n+1) + 7n$$

4. Infinite summations

4.1. **Telescoping where the one end has a finite limit.** Consider the infinite series summation:

$$\sum_{n=1}^{\infty} g(n)$$

Suppose $g = \Delta f$ for some function f, and $\lim_{n\to\infty} f(n) = L$. Then the above summation is $L - f(1)$. For instance, consider:

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$$

As already discussed, $f(n) = -1/n$ here, and it limits to 0, so the summation is $0 - (-1) = 1$.

More generally:

$$\sum_{n=1}^{\infty} \frac{1}{n(n+m)} = \frac{1}{m}\sum_{k=1}^{m}\frac{1}{k}$$

Thus, for instance:

$$\sum_{n=1}^{\infty} \frac{1}{n(n+3)} = \frac{1}{3}\left[1 + \frac{1}{2} + \frac{1}{3}\right] = \frac{11}{18}$$

Infinite series sums are to summations of finitely many terms what improper integrals are to proper integrals.

4.2. **When does a series converge?** For now, we restrict attention to series all of whose terms are nonnegative. Thus, the sequence of partial sums is non-decreasing. We say that a series *converges* if the sum of the series is finite, and it *diverges* if the sum is $+\infty$. For series of nonnegative terms, these are the only possibilities.

For most situations, we can do one of the following four things:

(1) Show that the series diverges.
(2) Show that the series converges, and find its sum.
(3) Show that the series converges, and find bounds on its sum, but find no summation-free expression for the infinite series sum.
(4) Show that the series converges, without any explicit bounds on its sum.

Notice that in cases (2)-(4), we have shown that the series converges, but our degree of understanding of the sum differs. While (2) is the most desirable, (3) is great too and even (4) is often good. Finite numbers may look very different from each other, but they're a lot smaller than ∞.

We are now ready to give a bunch of results about series summations. Note that when I say *term* of a series, I mean the summand, and when I say *partial sum*, I mean the sum of an initial segment of the series.

(1) A series of nonnegative terms converges to the least upper bound of its sequence of partial sums (which is monotonic increasing). In particular, a series converges if and only if its sequence of partial sums has an upper bound, and any upper bound on the sequence of partial sums also serves as an upper bound on the sum of the series.
(2) If a series of nonnegative terms converges, the terms in the series must tend to 0. The contrapositive of this is: if the terms in a series of nonnegative terms do not go to zero, the series diverges. This criterion can be used to easily show, for many series, that they diverge. However, it is a *necessary but not sufficient* condition for convergence, and cannot be used to establish that any given series converges.
(3) Permuting the terms does not change either the convergence or the value of the sum of a series of nonnegative terms.
(4) Left shifts and/or changing finitely many terms does not change the convergence of a series though it may change the value of the sum of the series. (This last result also holds for series with negative terms).

5. Geometric series

5.1. **Geometric series plain and simple.** A *geometric series* with initial term a and common ratio r or *geometric progression* is a series described in the following equivalent ways:

(1) In terms of a *recursive relation*, where each term is r times the previous term. Explicitly, the relation is $a_n = ra_{n-1}$.
(2) In terms of a direct description of the n^{th} term, we have $a_n = r^{n-1}a$.

To make the indexing easier for geometric series, we often start the terms from 0 onward. In this case, if the 0^{th} term is $a = a_0$, then the n^{th} term is $r^n a_0$. To avoid degenerate cases, we assume $a \neq 0$ and $r \neq 0$.

The formula for the finite partial sum, for $r \neq 1$, is:

$$a + ar + ar^2 + \cdots + ar^n = \frac{a(1 - r^{n+1})}{1 - r} = \frac{a(r^{n+1} - 1)}{r - 1}$$

In the case $r = 1$, the sum is just $(n+1)a$.

The infinite series sum is given as follows:

(1) If $|r| < 1$, the sum is $\lim_{n\to\infty} a(1 - r^{n+1})/(1 - r)$, which becomes $a/(1 - r)$. In the subcase $0 < r < 1$, the series converges monotonically. In the subcase $-1 < r < 0$, the series converges, but not monotonically, because the term signs are alternating.
(2) If $r = 1$, the terms of the series are constant, and the partial sums are just $(n+1)a$, which goes to $+\infty$ or $-\infty$ depending on the sign of a.
(3) If $r = -1$, the terms of the series oscillate between two finite numbers, namely a and 0.
(4) If $r > 1$, the series diverges monotonically to $+\infty$ or $-\infty$, depending on the sign of a.

(5) If $r < -1$, the series has oscillatory divergence – the magnitude of the terms gets larger, but the sign keeps oscillating.

Also of interest is the notion of *eventually geometric series*. An eventually geometric series is a series whose terms eventually resemble those of a geometric series. The sum of an eventually geometric series can be computed as follows: deal with the initial few, anomalous terms, separately by adding them up, and use the series summation formula for the remaining infinitely many terms.

For instance, consider the series:

$$5 + 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \dots$$

Here, the first term is anomalous, but we observe that the remainder of the series has initial term 1 and common ratio $1/2$. The sum of the eventually geometric part is thus $1/(1-(1/2)) = 2$. The total sum of the series is thus $5 + 2 = 7$.

5.2. **Geometric series as infinite expansions.** We have noted that, for $|r| < 1$, we have:

$$1 + r + r^2 + r^3 + \dots = \frac{1}{1-r}$$

Replacing the variable r by the letter x, we get:

$$\frac{1}{1-x} = 1 + x + x^2 + \dots = \sum_{k=0}^{\infty} x^k$$

with the expansion being valid for $|x| < 1$.

This allows us to expand out $1/(1+ax)$ in terms of a geometric series. Specifically, for $a \neq 0$:

$$\frac{1}{1+ax} = \sum_{k=0}^{\infty} (-a)^k x^k$$

with the expansion valid for $|x| < 1/|a|$.

Similarly, we can expand:

$$\frac{x}{1+ax^2} = x \sum_{k=0}^{\infty} (-ax^2)^k = \sum_{k=0}^{\infty} (-a)^k x^{2k+1}$$

where $|x| < 1/\sqrt{|a|}$.

Converting a compactly expressed rational function in terms of an infinite series may seem a little stupid. But there are various things we can do with infinite *power series* – they are infinite analogues of polynomials. With suitable caveats, we can perform term-wise integration and differentiation on the series. We shall return to power series a little later in the course.

5.3. **Like a geometric series with two common ratios.** Consider the series:

$$1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \frac{1}{36} + \dots$$

This is like a geometric series, with two common ratios – the ratios alternate between $1/2$ and $1/3$. There are several ways of handling this. One is to group together adjacent pairs of terms, and get:

$$\frac{3}{2} + \frac{1}{4} + \frac{1}{24} + \dots$$

We see now that this is a genuine geometric series with initial term $3/2$ and common ratio $1/6$. The infinite series sum is thus $(3/2)/(1-(1/6)) = 9/5$.

Another way is to split the geometric series into two sub-series:

$$1 + \frac{1}{6} + \frac{1}{36} + \dots$$

and:

$$\frac{1}{2} + \frac{1}{12} + \frac{1}{72} + \dots$$

Both are geometric series, with sums 6/5 and 3/5 respectively, and the total sum is 9/5 (we can do this splitting and rearrangement with impunity because we are working with a series of positive terms).

5.4. **Summability versus integrability of geometric series.** We noted above that a geometric series is summable to infinity if the common ratio is less than 1. The analogous observation with integrals is that:

$$\int_0^\infty a^x \, dx$$

is finite if $a < 1$ and infinite if $a \geq 1$. Note that the indefinite integral is $a^x/(\ln a)$. If $a < 1$, then the integral from 0 to ∞ is $-1/(\ln a)$.

6. Relating summations to integrals: the integral test

6.1. **The integral test: numerical relationship.** Suppose f is a continuous non-increasing nonnegative function defined on $[1, \infty)$ (though possibly on more real numbers). Suppose that $\lim_{x \to \infty} f(x) = 0$. We can consider a discrete and continuous integral of f:

$$\sum_{n=1}^\infty f(n)$$

versus:

$$\int_1^\infty f(x) \, dx$$

We can verify both geometrically and algebraically that:

$$f(n) \geq \int_n^{n+1} f(x) \, dx \geq f(n+1)$$

Summing up over all $n \in \mathbb{N}$, we obtain:

$$\sum_{n=1}^\infty f(n) \geq \int_1^\infty f(x) \, dx \geq \sum_{n=2}^\infty f(n)$$

The last term is the full summation minus $f(1)$, and we get:

$$\sum_{n=1}^\infty f(n) \geq \int_1^\infty f(x) \, dx \geq -f(1) + \sum_{n=1}^\infty f(n)$$

Equivalently:

$$\int_1^\infty f(x) \, dx \leq \sum_{n=1}^\infty f(n) \leq f(1) + \int_1^\infty f(x) \, dx$$

Thus, the infinite series sum and the infinite integral are bounded in terms of each other in a very precise sense. In particular, this implies that the summation is finite if and only if the integral is finite. Moreover, we can use the value of one of them to estimate the other one.

Consider, for instance:

$$\sum_{n=1}^\infty \frac{1}{n^2}$$

Note that $f(x) = 1/x^2$ is a continuous decreasing function with $\lim_{x \to \infty} f(x) = 0$. Thus, we can apply the above idea. We first calculate the definite integral $\int_1^\infty dx/x^2$, which turns out to be 1. We thus get:

$$1 \leq \sum_{n=1}^{\infty} \frac{1}{n^2} \leq 1 + 1$$

Thus, the infinite series sum is finite, and between 1 and 2. What if we want a more refined estimate? There is a slight generalization of the above, which says that:

$$\sum_{n=1}^{M-1} f(n) + \int_M^{\infty} f(x)\,dx \leq \sum_{n=1}^{\infty} f(n) \leq \sum_{n=1}^{M} f(n) + \int_M^{\infty} f(x)\,dx$$

Setting $M = 3$ for $f(x) = 1/x^2$ gives:

$$\frac{19}{12} \leq \sum_{n=1}^{\infty} \frac{1}{n^2} \leq \frac{61}{36}$$

Thus, the summation is somewhere between 1.6 and 1.7. This already gives a very good bound, and we can refine the bound much further if we so desire.

Note: The actual value of $\sum_{n=1}^{\infty} 1/n^2$ is $\pi^2/6$, but we don't have the tools to obtain that answer yet. Many other such summations lack simple expressions for their sums.

6.2. **Eventual formulations of the integral test.** The integral test can be weakened somewhat if we are interested only in talking about convergence and are not interested in the actual value of the integral.

The pure form of the integral test requires f to be continuous, nonnegative, and non-increasing. However, we can modify this to requiring that f *eventually* satisfy these conditions – the behavior of f in the beginning does not matter. Further, we can calculate the integral $\int_a^{\infty} f(x)\,dx$ from any finite number a beyond which f starts behaving nicely. If this integral is bounded, then f sums to a finite number. The reason is that the first few terms of f are only finitely many, and throwing them in or out does not affect whether the sum is convergent.

Note that when we shift to the *eventually* formulation, then we lose out on something: the concrete numerical bound. We can retrieve this relationship, but we need to do more work. As pointed earlier, though, even knowing that something converges is useful information.

6.3. **p-series and ζ-functions.** For any $p > 0$ (*not necessarily an integer*), consider the following series, called the p-series:

$$\zeta(p) := \sum_{n=1}^{\infty} \frac{1}{n^p}$$

We apply the integral test. We see that:

$$\int_1^{\infty} \frac{dx}{x^p} = \begin{cases} \frac{1}{p-1}, & p > 1 \\ \infty, & 0 < p \leq 1 \end{cases}$$

We thus see that the summation is infinite when $0 < p \leq 1$. Let's consider the case that $p > 1$. In this case, we see that the summation is bounded between $1/(p-1)$ and $p/(p-1)$ (an interval of length 1). In symbols, ζ is a function on $(1, \infty)$ satisfying:

$$\frac{1}{p-1} \leq \zeta(p) \leq \frac{p}{p-1}$$

We can also see the following things with some reflection:

(1) $\lim_{p \to 1^+} \zeta(p) = \infty$. We can see this from the fact that $\lim_{p \to 1^+} 1/(p-1) = \infty$.
(2) ζ is a continuous decreasing function of p on $(1, \infty)$, and $1/(p-1) < \zeta(p) < p/(p-1)$ for all $p > 1$.
(3) $\lim_{p \to \infty} \zeta(p) = 1$. Thus, $y = 1$ is a horizontal asymptote for ζ.

It turns out that $\zeta(2)$ (which, a little while ago, we bounded between 1.6 and 1.7) actually takes the value $\pi^2/6$, which is between 1.64 and 1.65. Arriving at this concrete expression requires plenty of effort, that we shall not undertake. In a similar vein, we can compute $\zeta(4)$, which turns out to be $\pi^4/90$. In fact, $\zeta(2n)$ is a rational multiple of π^{2n} for any natural number n. The ζ-values for odd numbers do not have known expressions. In 1978, somebody proved that $\zeta(3)$ is irrational, and many questions about $\zeta(3)$ are still unresolved.

7. Two other theorems

7.1. **The basic comparison theorem.** This corresponds to Theorem 12.3.6. The book's formulation has a typographical error and the correct version is stated here.

Suppose we have two series $\sum a_k$ and $\sum b_k$, both with nonnegative terms. Then, if there exists some k_0 such that:

$$a_k \leq b_k \ \forall k \geq k_0$$

This implies that:

(1) If $\sum b_k$ converges, so does $\sum a_k$.
(2) If $\sum a_k$ diverges, so does $\sum b_k$.

A few comments will make clear what is happening. First, note that if $a_k \leq b_k$ for *every* k (i.e., $k_0 = 1$) then each partial sum of the a-series is bounded by the corresponding partial sum of the b-series. Since the latter sequence of partial sums converges, the former is a bounded non-decreasing sequence and hence must converge to its least upper bound. This explains part (1) in the case $k_0 = 1$.

What happens if k_0 is something other than 1? Essentially the same proof works, once we throw out the first few terms. Basically, what matters is not *complete domination*, but *eventual domination*.

But we do lose something. Specifically, it is no longer true that the infinite series sum of the a_ks is smaller than the infinite series sum of the b_ks. This is because the first few values of the a_ks could be really large. Thus, we again sacrifice a *numerical relation* when we move from a universal constraint to its corresponding eventual constraint, while we still preserve *whether or not convergence occurs*.

7.2. **The limit comparison theorem.** This states that if we have two series $\sum a_k$ and $\sum b_k$, both of which have *positive terms only*, then if the sequence of quotients a_k/b_k approaches a positive number L, we have that $\sum a_k$ converges if and only if $\sum b_k$ converges.

What's going on here? If the quotient approaches a positive number, each sequence is bounded by a constant multiple of the other sequence. The proof, as given in the book, uses a mild ϵ-argument, and it is worth going through.

8. Degree difference rules

8.1. **When do summations of rational function series converge?** Suppose we are given a series whose general term is a rational function. Assume that the denominator of the rational function does not blow up anywhere. How do we determine whether the series converges? The following simple criterion works:

> If the degree of the denominator minus the degree of the numerator is *strictly greater* than 1, then the series converges. If the degree of the denominator minus the degree of the numerator is equal to or less than 1, then the series diverges.

The key idea is as follows: if $f(x)/g(x)$ is a rational function and the degree of g minus the degree of f is p, then we can apply the limit comparison theorem between the two series $a_n = f(n)/g(n)$ and $b_n = 1/n^p$. The quotient limits to a finite nonzero number, which is basically given by the quotient of the leading coefficients of f and g. For the latter series, we know by the integral test that it converges iff $p > 1$. The result therefore extends to the former series as well.

Note that the terms of the series for a rational function need not be positive from the get-go. However, they will *eventually* have constant sign. If they are eventually negative, pull out a minus sign and we are then in a position to use the limit comparison theorem.

8.2. **An example: using integral test directly versus degree difference rule.** For instance, consider the summation:

$$\sum_{n=1}^{\infty} \frac{n}{n^2+1}$$

First, we note that the series terms are continuous and nonnegative, and because the denominator has higher degree than the numerator, the terms are eventually non-increasing. In fact, in this case, they are non-increasing right from the beginning, but this is not obvious.

We carry out the corresponding integration:

$$\int_1^\infty \frac{x\,dx}{x^2+1}$$

The integral is:

$$\left[\frac{1}{2}\ln(x^2+1)\right]_1^\infty$$

which is ∞. Thus the corresponding summation is infinite.

We could, however, also use the degree difference rule directly to conclude tht the series diverges: the degree of denominator minus degree of numerator is equal to 1, so the series diverges.

Similarly, consider the summation:

$$\sum_1^\infty \frac{1}{n^2+1}$$

Again, the series satisfies the conditions for th integral test. Thus, we carry out the corresponding integration:

$$\int_1^\infty \frac{dx}{x^2+1} = [\arctan x]_1^\infty = (\pi/2) - (\pi/4) = \pi/4$$

We can use the concrete version of the integral test to show that the summation is thus bounded between $\pi/4$ (about 0.785) and $(1/2) + (\pi/4)$ (about 1.285).[1]

Alternatively, instead of the integral test, we could use the degree difference rule directly and conclude that the summation converges.

8.3. **A rough power calculation.** The rule above works heuristically even when, instead of rational functions, we throw in fractional powers, logarithms, and other stuff. The general rule is that logarithms count for roughly a power of 0 (so they don't affect the degree calculations for either the numerator or the denominator), but they could play a role of tie-breaker. Another way of thinking of this is that $\ln x$ is roughly x^{0^+}. Here's one way of putting it:

(1) If the denominator degree minus the numerator degree is strictly greater than 1, the summation converges. The classic examples are p-series.
(2) If the denominator degree minus the numerator degree is strictly less than 1, the summation diverges.
(3) If the denominator degree minus the numerator degree is exactly 1, then the series could converge or diverge. We need to use the integral test to determine what is really happening. Even within this, we can use the following: if the denominator degree minus the numerator degree is 1^-, then the series diverges. If it is 1^+, then we need to do more work.

Let us elaborate on point (3) above. Consider the series:

$$\sum_{n=1}^\infty \frac{\ln n}{n}$$

Here, ln is roughly like n^{0^+}, so the degree of the denominator minus the degree of the numerator is like $1 - (0^+) = 1^-$. Hence, we can be sure that this diverges. On the other hand, for the series $\sum_{n=1}^\infty \frac{1}{n(1+\ln n)}$ and $\sum_{n=1}^\infty \frac{1}{n(1+(\ln n)^2)}$, the rule is inconclusive, so we need to use the integral test.

Consider:

[1] The actual summation turns out to be about 1.08.

$$\sum_{n=1}^{\infty} \frac{1}{n(1+\ln n)}$$

This is continuous, non-negative, and non-increasing, so we can use the integral test. The coresponding integration is:

$$\int_{1}^{\infty} \frac{dx}{x(1+\ln x)}$$

Solving the integration, we get:

$$[\ln(1+\ln x)]_1^{\infty}$$

which diverges. Hence, so does the summation.

On the other hand, for the series:

$$\sum_{n=1}^{\infty} \frac{1}{n(1+(\ln n)^2)}$$

the corresponding integration is:

$$\int_{1}^{\infty} \frac{dx}{x(1+(\ln x)^2)}$$

This simplifies to:

$$[\arctan(\ln x)]_1^{\infty}$$

which is a finite number, namely $\pi/2$.

Hence, the summation converges. However, we do not know *a priori* what it converges to. With a little effort, we could try to find bound for the summation via the concrete version of the integral test.