REVIEW SHEET FOR MIDTERM 2: ADVANCED

MATH 153, SECTION 55 (VIPUL NAIK)

To maximize efficiency, please bring a copy (print or readable electronic) of both the basic and the advanced review sheet AND the previous review sheet to the review session.

In addition to spotting the errors in the existing reasoning, try to solve the problems and find the *correct* solution for each of the error-spotting exercises.

1. Left-overs from integration

1.1. Integrating radicals.

Error-spotting exercises..

(1) Consider the integration:

$$\int_0^1 \sqrt{1-x^2}\,dx$$

To perform this integration, we put $\theta = \arcsin x$. Then $x = \sin\theta$ and $\sqrt{1-x^2} = \cos\theta$, so we get:

$$\int_0^1 \cos\theta\,d\theta$$

This simplifies to $\sin 1 - \sin 0 = \sin 1$.

(2) Consider the integral

$$\int_0^\infty \frac{dx}{(x^2+a^2)^{3/2}}$$

Let $\theta = \arctan(x/a)$. Then, the above becomes:

$$\int_0^{\pi/2} \frac{\sec^2\theta\,d\theta}{a^3\sec^3\theta}$$

This simplifies to:

$$\frac{1}{a^3}\int_0^{\pi/2} \cos\theta\,d\theta$$

The integral of cos gives the value 1, so the overall answer is $1/a^3$.

(3) Consider the integral:

$$\int_0^\infty \frac{dx}{x^2+2x\cos\alpha+1}$$

Completing the square in the denominator, we get:

$$\int_0^\infty \frac{dx}{(x+\cos\alpha)^2+\sin^2\alpha}$$

We thus get:

$$\left[\frac{1}{\cos\alpha}\arctan\left(\frac{x+\cos\alpha}{\sin\alpha}\right)\right]_0^\infty$$

Plugging in limits and evaluating, we get:

$$\frac{1}{\cos\alpha}(\pi/2-\cot\alpha)$$

This simplifies to:

$$\frac{\pi}{2\cos\alpha} - \csc\alpha$$

1.2. **Partial fractions.** Error-spotting exercises:

(1) Suppose a, b are distinct positive numbers. We can do the integration:

$$\int \frac{dx}{(x^2+a^2)(x^2+b^2)}$$

as follows:

$$\int \frac{dx}{(x^2+a^2)(x^2+b^2)} = \frac{1}{a^2-b^2}\int \frac{(x^2+a^2)-(x^2+b^2)}{(x^2+a^2)(x^2+b^2)}\,dx$$

This simplifies to:

$$\frac{1}{a^2-b^2}\int \left[\frac{1}{x^2+a^2} - \frac{1}{x^2+b^2}\right] dx$$

This becomes:

$$\frac{1}{a^2-b^2}[\ln(x^2+a^2) - \ln(x^2+b^2)] + C$$

(2) Consider the integration:

$$\int_{-\infty}^{\infty} \frac{dx}{(x^2+a^2)^2}$$

The integration gives:

$$\left[\frac{-1}{x^2+a^2}\right]_{-\infty}^{\infty}$$

Both the endpoint limits are zero, so the integral is 0.

(3) Consider the integral

$$\int \frac{dx}{(x-\alpha)(x-\beta)(x-\gamma)}$$

where α, β, γ are distinct real numbers.
Using partial fractions, we get that the integral is:

$$\int \frac{(\alpha-\beta)(\alpha-\gamma)\,dx}{x-\alpha} + \frac{(\beta-\gamma)(\beta-\alpha)\,dx}{x-\beta} + \frac{(\gamma-\alpha)(\gamma-\beta)\,dx}{x-\gamma}$$

This gives:

$$(\alpha-\beta)(\alpha-\gamma)\ln|x-\alpha| + (\beta-\gamma)(\beta-\alpha)\ln|x-\beta| + (\gamma-\alpha)(\gamma-\beta)\ln|x-\gamma| + C$$

1.3. **Improper integrals.** Error-spotting exercises:

(1) We have:

$$\int_{-\infty}^{\infty} \frac{dx}{x^3} = \left[\frac{1}{2x^2}\right]_{-\infty}^{\infty} = 0$$

(2) We have:

$$\int_{-\infty}^{\infty} \frac{x\,dx}{x^2+1} = \lim_{a\to\infty}\int_{-a}^{a} \frac{x\,dx}{x^2+1} = \lim_{a\to\infty} 0 = 0$$

where the integral on any finite interval $[-a, a]$ is zero on account of the function being an odd function.

2. Differential equations

2.1. Solving differential equations at large. Error-spotting exercises

(1) Consider the differential equation:

$$(y')^2 - 3yy' + 2y^2 = 0$$

his factors as:

$$(y' - y)(y' - 2y) = 0$$

Thus, either $y' = y$ or $y' = 2y$ for which respective general solutions are Ce^x and Ce^{2x}. The general solution is thus of the form $C_1 e^x + C_2 e^{2x}$ where C_1, C_2 are arbitrary real numbers.

(2) Consider the differential equation:

$$\frac{dy}{dt} = \sqrt{y(1-y)}$$

Solving, we get:

$$y = \frac{1}{2}(1 + \sin(t + C))$$

The solution to this differential equation thus looks like a sinusoidal oscillating curve.

2.2. Graphical interpretation and initial value problems. Words ...

(1) Consider the differential equation:

$$\frac{dy}{dt} = y \ln y$$

with the additional condition that $y(0) > 0$. The solution to this differential equation is a function y of t such that $\lim_{t\to\infty} y(t) = \infty$ and the function grows doubly exponentially.

(2) Consider the differential equation:

$$\frac{dy}{dt} = y^2$$

with $y(0) = 1$. Then, solving the differential equation, we get:

$$\frac{-1}{y} = t + C$$

Plug in $y(0) = 1$ to get $C = -1$, so we get:

$$y = \frac{1}{1-t}$$

The upshot is that $\lim_{t\to\infty} y(t) = 0$.

3. The least upper bound axiom

Error-spotting exercises ...

(1) Suppose T is a nonempty subset of a nonempty set S. Then, the glb of T is less than or equal to the glb of S. Similarly, the lub of T is less than or equal to the lub of S.

(2) For subsets A and B of $\mathbb{R}$, denote by AB the set $\{ab \mid a \in A, b \in B\}$. Then, if A and B are both nonempty and bounded, so is AB. Further, the glb of AB is the product of the glb of A and the glb of B. Similarly, the lub of AB is the product of the lub of A and the lub of B.

4. Sequences of reals

4.1. Sequences: basics. No error-spotting exercises

4.2. Continuous-discrete interplay. No error-spotting exercises

5. Limit computation techniques

Error-spotting exercises ...

(1) Consider the limit computation problem:

$$\lim_{x \to 0} \frac{2 - \cos x}{x^2}$$

To do this limit, we apply the LH rule once to get:

$$\lim_{x \to 0} \frac{\sin x}{x}$$

Apply the LH rule another time and get:

$$\lim_{x \to 0} \frac{\cos x}{1}$$

This evaluates to 1.

(2) Consider the limit computation problem:

$$\lim_{x \to 0} \frac{x - \sin x}{(1 - \cos x)\ln(1 + x)}$$

The numerator has a zero of order three, and the denominator is a product of terms of zeros of order two and one. Hence the denomintor has a zero of order three as well. Thus, the quotient has a limiting value of 1.

(3) Consider the limit computation problem:

$$\lim_{x \to 1} \frac{\sin(2\pi x)}{\sin(\pi x)}$$

We can strip off the sin from the numerator and denominator and we get:

$$\lim_{x \to 1} \frac{2\pi x}{\pi x}$$

This simplifies to 2.

(4) Consider the limit problem:

$$\lim_{x \to \pi/2^-} \frac{\tan(5x)}{\tan x}$$

We can strip off the tan from both the numerator and the denominator, and we are left with a limit of 5.

6. Tricky topics

6.1. **Important classes of functions outside of the elementary world.** *This is not part of the executive summary of any of the lecture notes, but is related to some of the homework problems and the challenge problem.*

We note here three such classes of functions:

(1) The function arising as an integral of $\exp(-x^2)$. All the following functions have antiderivatives expressible in terms of this antiderivative and elementary functions:
- $x^{2k}\exp(-mx^2)$ where k is a positive integer and m is a positive real: The idea here is to use integration by parts, taking x^{2k-1} as the part to differentiate and $x\exp(-mx^2)$ as the part to integrate. The problem reduces to $x^{2k-2}\exp(-mx^2)$. Repeat.
- $x^{m+(1/2)}\exp(-x)$ where m is an integer (not necessarily positive. Putting $u = \sqrt{x}$ allows us to go back and forth with $\exp(-x^2)$. Note that the $x^{m+(1/2)}\exp(-x)$ are related to each other for different values of m via integration by parts: take $\exp(-x)$ as the part to integrate.
- $\exp(x^{-2/(2m+1)}$ where m is a positive integer. Put $u = x^{1/(2m+1)}$ to go back and forth.

In particular, in applicable cases, we can calculate the *improper definite integral* on all of $\mathbb{R}$ for each of these functions based on the fact that:

$$\int_{-\infty}^{\infty} \exp(-x^2) = \sqrt{\pi}$$

In particular, since the function is even, we have:

$$\int_{0}^{\infty} \exp(-x^2) = \frac{\sqrt{\pi}}{2}$$

We define:

$$\Gamma(x) := \int_0^\infty t^{x-1}e^{-t}\, dt$$

for $x \in (0, \infty)$.

Based on the fact about the integral of $\exp(-x^2)$, we can calculate the value of Γ at all half-integers. Note that for x a positive integer, $\Gamma(x) = (x-1)!$ as seen in a homework exercise.

(2) The function arising as an integral of $1/(\ln x)$, for $x > 1$. All the following have antiderivatives expressible in terms of this antiderivatve and elementary functions:
- $\ln(\ln x)$ for $x > 1$.
- e^x/x, for $x > 0$. Also, $e^x \ln x$ and e^x/x^r for all positive integers r.
- e^{e^x} for all x.

(3) The function arising as an integral of $(\sin x)/x$ (we fill in the value of the integrand at 0 to be 1). All the following have antiderivatives expressible in terms of this: $(\cos x)(\ln x)$, $(1-\cos x)/x^2$, $(x - \sin x)/x^3$. The back-and-forth technique is integration by parts: one thing gets integrated, the other differentiated.

In particular, given that:

$$\int_0^\infty \frac{\sin x}{x}\, dx = \frac{\pi}{2}$$

we can compute the integrals from 0 to ∞ of $(1-\cos x)/x^2$ and $(x-\sin x)/x^3$.

7. Quickly

This section lists things you should be able to do quickly.

7.1. **Sequence pattern finding.** You should be able to do the following:

(1) Given the first few terms of an easy sequence, predict the next few terms.
(2) Describe a sequence using a recurrence relation.
(3) Write the general expression (possibly making cases if there's periodic-like behavior) for the n^{th} term.
(4) Move back and forth between general term descriptions and recurrence relation descriptions of sequences.

7.2. **Our common values.** Preferably remember these (or be capable of computing quickly) to at least one digit. Generally, you will *not* be asked to do any numerical computations using these. In practice, the main way this is useful is to figure out whether something is positive or negative. For instance, is $3\sqrt{2} - 4$ positive? What about $e^2 - 8$? Often, there are other ways of answering such questions, but remembering the numerical values is a quick and dirty approach.

(1) Square roots of 2, 3, 5, 6, 7, 10.
(2) Natural logarithms of 2, 3, 5, 7, and 10.
(3) Value of π, $1/\pi$, $\sqrt{\pi}$, and π^2.
(4) Value of e, $1/e$.
(5) Some relative logarithms, such as $\log_2 3$ or $\log_2(10)$. Although you don't need these values to a significant degree of precision, it is useful to have some idea of their magnitude.

7.3. **Adding things up: arithmetic.** You should be able to:

(1) Do quick arithmetic involving fractions.
(2) Sense when an expression will simplify to 0.
(3) Compute approximate values for square roots of small numbers, π and its multiples, etc., so that you are able to figure out, for instance, whether $\pi/4$ is smaller or bigger than 1, or two integers such that $\sqrt{39}$ is between them.
(4) Know or quickly compute small powers of small positive integers. This is particularly important for computing definite integrals. For instance, to compute $\int_2^3 (x+1)^3\,dx$, you need to know/compute 3^4 and 4^4.

7.4. **Computational algebra.** You should be able to:

(1) Add, subtract, and multiply polynomials.
(2) Factorize quadratics or determine that the quadratic cannot be factorized.
(3) Factorize a cubic if at least one of its factors is a small and easy-to-spot number such as 0, ± 1, ± 2, ± 3. *This could be an area for potential improvement for many people.*
(4) Factorize an even polynomial of degree four. *This could be an area for potential improvement for many people.*
(5) Do polynomial long division.
(6) Solve simple inequalities involving polynomial and rational functions once you've obtained them in factored form.

7.5. **Computational trigonometry.** You should be able to:

(1) Determine the values of sin, cos, and tan at multiples of $\pi/2$.
(2) Determine the intervals where sin and cos are positive and negative.
(3) Remember the formulas for $\sin(\pi \pm x)$ and $\cos(\pi \pm x)$, as well as formulas for $\sin(-x)$ and $\cos(-x)$.
(4) Recall the values of sin and cos at $\pi/6$, $\pi/4$, and $\pi/3$, as well as at the corresponding obtuse angles or other larger angles.
(5) Reverse lookup for these, for instance, you should quickly identify the acute angle whose sin is $1/2$.
(6) Formulas for double angles, half angles: $\sin(2x)$, $\cos(2x)$ in terms of sin and cos; also the reverse: $\sin^2 x$ and $\cos^2 x$ in terms of $\cos(2x)$.
(7) Remember the formulas for $\sin(A+B)$, $\cos(A+B)$, $\sin(A-B)$, and $\cos(A-B)$.
(8) Convert between products of sin and cos functions and their sums: for instance, the identity $2\sin A \cos B = \sin(A+B) + \sin(A-B)$. You don't have to remember these identities separately since they follow from the identities covered in the previous point, but you should be comfortable going back and forth.

7.6. **Computational limits.** This includes:

(1) Look at a rational function and determine its limit to $+\infty$ and $-\infty$ based on heuristics, without thought.
(2) Look at something involving a quotient of an exponential and a polynomial, or a polynomial and a logarithmic, and predict the behavior as we go to 0 or ∞.
(3) Know which functions can be stripped from composites.
(4) Apply the LH rule mentally multiple times and figure out how many times it needs to be applied.

7.7. **Computational differentiation.** You should be able to:

(1) Differentiate a polynomial (written in expanded form) once or twice on sight, without rough work.
(2) Differentiate sums of powers of x on sight (without rough work).
(3) Differentiate rational functions with a little thought.
(4) Do multiple differentiations of expressions whose derivative cycle is periodic, e.g., $a\sin x + b\cos x$ or $a\exp(-x)$.
(5) Do multiple differentiations of expressions whose derivative cycle is periodic up to constant factors, e.g. $a\exp(mx+b)$ or $a\sin(mx+\varphi)$.
(6) Differentiate simple composites without rough work (e.g., $\sin(x^3)$).
(7) Differentiate ln, exp, and expressions of the form f^g and $\log_f(g)$.

7.8. **Computational integration.** You should be able to:

(1) Compute the indefinite integral of a polynomial (written in expanded form) on sight without rough work.
(2) Compute the definite integral of a polynomial with very few terms within manageable limits quickly.
(3) Compute the indefinite integral of a simple-looking rational function and also the definite integral, with a little time, including improper integrals.
(4) Compute the indefinite integral of a sum of power functions quickly.
(5) Know that the integral of sine or cosine on any quadrant is ± 1.
(6) Compute the integral of $x \mapsto f(mx)$ if you know how to integrate f. In particular, integrate things like $(a + bx)^m$.
(7) Integrate sin, cos, $\sin^2$, $\cos^2$, $\tan^2$, $\sec^2$, $\cot^2$, $\csc^2$, $\sin^3$, $\cos^3$, $\tan^3$, $\sec^3$, $\cot^3$, $\csc^3$, and other higher powers of the basic trigonometric functions.
(8) Integrate on sight things such as $x \sin(x^2)$, getting the constants right without much effort.
(9) *By parts*: Integrate $\int x f(x)\,dx$ on sight if it is easy to integrate f twice, without having to think much (the answer is $x \int f - \int \int f$). Similarly, do $\int x^2 f(x)\,dx$ if you know how to integrate f twice.
(10) Using the previous point, integrate $\int f(\sqrt{x})\,dx$ with minimal stress and effort.
(11) *By parts*: Integrate $\int f(x)\,dx$ on sight if $xf'(x)$ is easy to integrate, e.g., $\int \ln x\,dx$.
(12) Remember the integrals for formats such as $e^x \cos x$ and $e^x \sin x$.
(13) If there is an easy choice of f such that $f + f' = g$, integrate $\int e^x g(x)\,dx = e^x f(x)$ on sight and similarly integrate $\int g(\ln x)\,dx = x f(\ln x)$ on siht.

7.9. **Being observant.** You should be able to look at a function and:

(1) Sense if it is odd (even if nobody pointedly asks you whether it is).
(2) Sense if it is even (even if nobody asks you whether it is).
(3) Sense if it is periodic and find the period (even if nobody asks you about the period).

7.10. **Graphing.** You should be able to:

(1) Mentally graph a linear function.
(2) Mentally graph a power function x^r (see the list of things to remember about power functions). Sample cases for r: 1/3, 2/3, 4/3, 5/3, 1/2, 1, 2, 3, -1, $-1/3$ $-2/3$.
(3) Graph a piecewise linear function with some thought.
(4) Mentally graph a quadratic function (very approximately) – figure out conditions under which it crosses the axis etc.
(5) Graph a cubic function after ascertaining which of the cases for the cubic it falls under.
(6) Mentally graph sin and cos, as well as functions of the $A \sin(mx)$ and $A \cos(mx)$.
(7) Graph a function of the form linear + trigonometric, after doing some quick checking on the derivative.
(8) Graph the inverse trigonometric functions arctan, arcsin, and arccos.

7.11. **Graphing: transformations.** Given the graph of f, you should be able to quickly graph the following:

(1) $f(mx)$, $f(mx + b)$: pre-composition with a linear function; how does $m < 0$ differ from $m > 0$?
(2) $Af(x) + C$: post-composition with a linear function, how does $A > 0$ differ from $A < 0$?
(3) $f(|x|)$, $|f(x)|$, $f(x^+)$, and $(f(x))^+$: pre- and post-composition with absolute value function and positive part fnuction.
(4) More slowly: $f(1/x)$, $1/f(x)$, $\ln(|f(x)|)$, $f(\ln |x|)$, $\exp(f(x))$, and other popular composites.