POWER SERIES AND CONVERGENCE ISSUES

MATH 153, SECTION 55 (VIPUL NAIK)

Corresponding material in the book: Section 12.8, 12.9.

What students should definitely get: The meaning of power series. The relation between Taylor series and power series, the notion of convergence, interval of convergence, and radius of convergence. The theorems for differentiation and integration of power series. Convergence at the boundary and Abel's theorem.

What students should hopefully get: How notions of power series provide an alternative interpretation for many of the things we have studied earlier in calculus. How results about power series combine with ideas like the root and ratio tests and facts about p-series.

Executive summary

Words ...

(1) The objects of interest here are power series, which are series of the form $\sum_{k=0}^{\infty} a_k x^k$. Note that for power series, we start by default with $k = 0$. If the set of values of the index of summation is not specified, assume that it starts from 0 and goes on to ∞. The exception is when the index of summation occurs in the denominator, or some other such thing that forces us to exclude $k = 0$.

(2) Also note that x^0 is shorthand for 1. When evaluating a power series at 0, we simply get a_0. *We do not actually do* 0^0.

(3) If a power series converges for c, it converges absolutely for all $|x| < |c|$. If a power series diverges for c, it diverges for all $|x| > |c|$.

(4) Given a power series $\sum a_k x^k$, the set of values where it converges is either 0, or $\mathbb{R}$, or an interval that (apart from the issue of inclusion of endpoints) is symmetric about 0. In particular, the interval could be of the following four forms: $(-c, c)$, $[-c, c]$, $[-c, c)$, and $(-c, c]$. The radius of convergence is c. Note that if the set of convergence is $\{0\}$, we say that the radius of convergence is 0, and if the power series converges everywhere, we say that the radius of convergence is ∞.

(5) Suppose a power series $\sum a_k x^k$ converges on an interval $(-c, c)$ to a function f. Then, f is infinitely differentiable on $(-c, c)$ and the power series for f' is obtained by differentiating the power series for f. In fact, the radius of convergence of the power series for f' is precisely the same as the radius of convergence of the power series for f. *On the other hand, the interval of convergence may differ* – the power series for f may converge at boundary points where the power series for f' does not. An example is arctan, which has interval of convergence $[-1, 1]$, but whose derivative has interval of convergence $(-1, 1)$.

(6) Suppose a power series $\sum a_k x^k$ converges on an interval $(-c, c)$ to a function f. Then, term wise integration of this power series gives an antiderivative of f on $(-c, c)$. In particular, if we choose the power series with constant term 0, we get the unique antiderivative that takes the value 0 at 0.

(7) Abel's theorem states that if $\sum a_k x^k = f(x)$ on $(-c, c)$, f is left continuous at c, and $\sum a_k c^k$ exists, then $f(c) = \sum a_k c^k$. Similarly, if f is right continuous at $-c$, and $\sum a_k (-c)^k$ exists, then $f(-c) = \sum a_k (-c)^k$.

(8) We can also consider power series centered at a: $\sum a_k (x - a)^k$. Everything translates nicely.

Deeper elaboration ...

(1) We have two kinds of operators: one from functions to power series (which involves taking the Taylor series) and the other from power series to functions (which involves summing up). It turns out that, if we start with a power series with a nonzero (possibly infinite) radius of convergence, look at the function it converges to, and take the Taylor series of that function, we retrieve the original power series. *This follows from the differentiation theorem stated above, which states that the derivative of a power series converges to the derivative of the function that the power series converges to.*

(2) On the other hand, it is possible to start with a function f infinitely differentiable on $\mathbb{R}$, take the Taylor series, and have the Taylor series converge to some function other than f. An example is the function that is e^{-1/x^2} for all $x \neq 0$ and 0 at $x = 0$. The function is infinitely differentiable everywhere and all its derivatives at 0 take the value 0. Thus, its Taylor series is 0, which obviously converges to the zero function rather than the specified function.

(3) It is also possible to have a function f that is infinitely differentiable on all of $\mathbb{R}$ such that the Taylor series of f converges to f, but the radius of convergence of the Taylor series is finite. More generally, it is possible that the interval of convergence of the Taylor series is smaller than the domain of the function. Two important examples in this direction are the arctan function (infinitely differentiable on all of $\mathbb{R}$ but interval of convergence $[-1, 1]$) and the function $\ln(1 + x)$ (infinitely differentiable on $(-1, \infty)$ but interval of convergence $(-1, 1]$).

(4) Call a function *globally analytic* if it is defined on all of $\mathbb{R}$ and has a power series about 0 that converges to the function everywhere. Sine, cosine, the exponential function, and polynomial functions are all globally analytic. Moreover, globally analytic functions are closed under addition, subtraction, multiplication, and composition.

(5) Call a function C^∞ on $\mathbb{R}$ if it is defined and infinitely differentiable on all of $\mathbb{R}$. The space of C^∞ functions is closed under addition, subtraction, multiplication, and composition. Moreover, any globally analytic function is C^∞. The converse is not true.

(6) A function is termed *analytic about* 0 if its Taylor series converges to it on an interval of nonzero radius. Any function that is analytic about 0 is infinitely differentiable (C^∞) around 0. However, the function may well be C^∞ on a bigger interval than the interval on which the Taylor series converges.

(7) If f and g have Taylor series that both converge on $(-c, c)$ to the respective functions, the Taylor series for $f + g$ also converges on $(-c, c)$ to it.

Actions ...

(1) We can consider functions in the following decreasing order of the behavior as $x \to \infty$: double exponential (like e^{e^x}, e^{2^x}), exponential in higher powers of x (e^{x^λ}, $\lambda > 1$), factorial ($x!$, $\Gamma(x)$, x^x), exponential (a^x, $a > 1$), exponential in lower powers of x (e^{x^λ}, $0 < \lambda < 1$), exponential in higher powers of $\ln x$ ($e^{(\ln x)^\lambda}$, $\lambda > 1$), polynomial or power functions of x (x^λ, which we can again split into cases based on whether $\lambda > 1$, $\lambda = 1$, or $0 < \lambda < 1$), polynomials in $\ln x$, $\ln(\ln x)$, and so on down.

(2) There is often quite a bit of separation within each level of the hierarchy (allowing for further stratification). Anything at a higher level in the hierarchy beats anything at a lower level in the hierarchy, so that the quotients tend to ∞ or 0 depending on which one is higher.

(3) The decay rates of the reciprocal functions mirror the growth rates of the functions.

(4) We use the term *superexponential* for functions that grow faster than exponential functions (for instance, e^{e^x}, e^{x^2}, and $x!$), *exponential* for functions that grow exponentially, and *subexponential* for functions that grow smaller than exponential.

(5) In general, when adding, subtracting, and multiplying, the larger one dominates. Division by a superexponential function leads to superexponential decay.

(6) Consider a power series $\sum a_k x^k$. If the a_k grow superexponentially in k, then the series converges only at 0. If the a_k decay superexponentially in k (i.e., $1/a_k$ grow superexponentially in k), then the series converges everywhere. [Justify to yourself using the ratio and/or root test]

(7) For $\sum a_k x^k$, if the a_k grow or decay exponentially, then the radius of convergence is finite and nonzero, and equals $\lim_{k\to\infty} 1/|a_k|^{1/k}$ – in other words, the reciprocal of the limiting common ratio of the a_ks. This is because at exponential growth, the a_ks match the x^ks and can affect the radius of convergence.

(8) For $\sum a_k x^k$, if the a_k grow or decay subexponentially, they have no effect on the radius of convergence – it is still 1. More generally, if a_k is the product of an exponential and a subexponential function, only the exponential function affects the radius of convergence. *The subexponential component does affect whether the endpoints are included in the interval of convergence.*

(9) As regards endpoints, the following is a rough statement. Consider $\sum a_k x^k$. If the a_ks are growing or constant, the series diverges at ± 1, so the interval of convergence is $(-1, 1)$. If the a_ks are decaying at a rate that is linear or slower, then the series does not absolutely converge, but it may conditionally

converge at one or both ends due to the alternating series theorem. If the a_ks are decaying at a rate that is $k^{-\lambda}, \lambda > 1$, then the series converges at both $+1$ and -1. Note that cases like $1/[k(1+(\ln k)^2)]$ are ambiguous, as discussed earlier.

(10) In particular, if $a_k = p(k)/q(k)$ where p and q are polynomials, the following can be said: if the degree of q is at least 2 greater than the degree of p, the interval of convergence is $[-1, 1]$. If the degree of q is equal to or less than the degree of p, the interval of convergence is $(-1, 1)$. If the degree of q is exactly one more than the degree of p, the interval of convergence is $[-1, 1)$. Note that the endpoint included may change under slight modifications of the situation, so you should also be aware of the reasoning process that leads to this conclusion. For instance, if there are only odd degree terms and nonnegative coefficients, we do not get any alternating series and the interval of convergence is $(-1, 1)$. On the other hand, if there are odd degree terms and alternating signs of coefficients among the odd degree terms, then the alternating series theorem applies at *both* ends -1 and 1.

1. Power series and convergence

So far, we have started with a function f and looked at the Taylor series for f, which is a power series – like a polynomial, except that the terms just keep going on. We now develop a general theory of power series, without being concerned about whether the power series arises as the Taylor series of a function. We define a power series as a series that looks like:

$$\sum_{k=0}^{\infty} a_k x^k$$

Note that we start the summation at 0*, and we interpret* x^0 *to be* 1*.*

For any particular value of x, we get a *series* in the usual sense – a series of numbers. Thus, for any particular value of $x \in \mathbb{R}$, we can ask whether this power series converges. We say that the power series converges on a set if it converges for all elements in the set.

We now state two basic results:

(1) If a power series converges for $x = c$, then it *converges absolutely* for all x satisfying $|x| < |c|$.
(2) If a power series diverges for $x = c$, then it diverges for all x satisfying $|x| > |c|$.

1.1. The interval of convergence.

From the above, we see the following possibilities for the set of values where a power series converges:

(1) Only 0: Note that a power series always converges at 0 to a_0. It is possible to have power series that do not converge anywhere else. What's happening is that the coefficients grow so fast that they overwhelm the geometric series x^k. For instance $\sum_{k=0}^{\infty} 2^{k^2} x^k$ diverges for all $x \neq 0$.
(2) Everywhere: This means that the power series converges for all real inputs. This happens when the coefficients go down so quickly as to overwhelm any geometric progression. Typical examples are the finite power series (i.e., polynomials) and the power series for sin, cos, and exp.
(3) There is a finite positive c such that the power series converges for all $|x| < |c|$ and diverges for all $|x| > |c|$. Such a c can be defined as the least upper bound over all $|x|$ where the series converges for x. This value c is termed the *radius of convergence*. Note that what happens *at* c and $-c$ is unclear. There are four possibilities for the interval of convergence: $[-c, c]$ (which means that convergence occurs both at c and at $-c$), $(-c, c)$ (which means that convergence occurs at neither endpoint), $[-c, c)$ and $(-c, c]$.

Here are examples for each of the four possibilities mentioned in (3): $\sum_{k=0}^{\infty} x^k$ converges only on $(-1, 1)$, $\sum_{k=1}^{\infty} x^k/k$ converges on $[-1, 1)$, $\sum_{k=1}^{\infty} (-1)^k x^k/k$ converges on $(-1, 1]$, and $\sum_{k=0}^{\infty} x^k/k^2$ converges on $[-1, 1]$.

Note that one remarkable thing about the interval of convergence is that, apart from the issue of inclusion of endpoints, it is symmetric about 0. This means that if we try to define the function:

$$f(x) = \sum_{k=0}^{\infty} a_k x^k$$

then the domain of f is almost symmetric about 0.

1.2. **Determining the radius of convergence.** We use the root test to determine the radius of convergence. Specifically, we note that if c is the radius of convergence, then for $|x| < c$, we expect $|a_n x^n|^{1/n}$ approaches something less than 1, and for $|x| > c$, it approaches something greater than 1. The radius of convergence should thus be chosen so that $|a_n c^n|^{1/n} \to 1$.

This gives us a formula for the radius of convergence:

$$c = \lim_{n \to \infty} \frac{1}{|a_n|^{1/n}}$$

if such a limit exists.

In case the limit on the right side is 0, then we are in case (1). If the limit in the denominator is 0, the radius of convergence is ∞.

There is a little problem with this, which is that some of the a_ns may be 0. The way to get around it is to use a notion of *limit superior* and *limit inferior* instead of limit. However, we have not developed these notions in detail, so will skip it. In practice, just take the limit for that subcollection of a_ns that are nonzero. This will suffice for most of the examples that we consider, though a foolproof approach must use the limit superior/limit inferior idea.

We can also adapt the ratio test to determine the radius of convergence. The adaptation of the ratio test yields that:

$$c = \lim_{n \to \infty} \frac{|a_n|}{|a_{n+1}|}$$

if such a limit exists. This works particularly well when the a_ns involve factorials.

2. Differentiation and integration of power series

Given a power series:

$$\sum_{k=0}^{\infty} a_k x^k$$

We can apply a procedure called *formal differentiation* or *term wise differentiation*, which basically just differentiates it term by term. We get:

$$\sum_{k=1}^{\infty} k a_k x^{k-1}$$

Re-indexing the dummy variable for summation, we get:

$$\sum_{k=0}^{\infty} (k+1) a_{k+1} x^k$$

This is also a power series.

We have the following results:

(1) If a power series converges on $(-c, c)$, the power series obtained by term wise differentiation also converges on $(-c, c)$ (Theorem 12.9.1). Note that this does *not* mean that wherever a power series converges, so does the formal derivative. It is possible that a power series converges on $[-c, c]$ but its derivative does not converge at one or both of the boundary points.
(2) *The differentiability theorem* (Theorem 12.9.2): If a power series converges on $(-c, c)$, then its formal derivative power series converges on $(-c, c)$, and the function to which the formal derivative converges is the derivative of the function to which the series converges. Formally, if $f(x) = \sum_{k=0}^{\infty} a_k x^k$, then $f'(x) = \sum_{k=1}^{\infty} k a_k x^{k-1}$.
(3) As a corollary of the differentiability theorem, if f is the function to which the power series $\sum_{k=0}^{\infty} a_k x^k$ converges on $(-c, c)$, then f is infinitely differentiable on $(-c, c)$ and each of its derivatives can be expressed using a power series obtained by differentiating the power series for f the required number of times.

(4) *Term by term integration* (Theorem 12.9.3): If $f(x) = \sum_{k=0}^{\infty} a_k x^k$ on $(-c, c)$, then term by term integration of the power series of f converges to an antiderivative of f on $(-c, c)$. The new series is $\sum_{k=0}^{\infty} a_k x^{k+1}/(k+1)$. Note that this is the particular antiderivative that takes the value 0 at 0. Adding a constant C gives the antiderivative that takes the value C at 0.

(5) *Abel's theorem* (Theorem 12.9.5): Suppose that $f(x) = \sum_{k=0}^{\infty} a_k x^k$ on $(-c, c)$. Then, if f is left continuous at c and $\sum_{k=0}^{\infty} a_k c^k$ converges, it converges to $f(c)$. Similarly, if f is right continuous at c and $\sum_{k=0}^{\infty} a_k (-c)^k$ converges, it converges to $f(-c)$.

In particular, when a power series converges everywhere, we can merrily differentiate and integrate to our hearts' contents. Combining these theorems, we can conclude that the radius of convergence of a power series and of its formal derivative are exactly equal, though it is possible that the power series converges at one or both endpoints for the function but not at the derivative..

3. Power series and Taylor series

3.1. **Back and forth.** We have done two kinds of things:

(1) Start from a (infinitely differentiable) function and compute the Taylor polynomials and Taylor series of the function. The goal is to approximate the function using polynomials.

(2) Start from a power series, figure out where it converges, and consider the function to which it converges. On the interior of the interval of convergence, the function is infinitely differentiable.

We have two kinds of mappings:

Infinitely differentiable functions $\rightsquigarrow$ Power series (i.e., the Taylor series of the function)

Power series $\rightsquigarrow$ Infinitely differentiable functions (i.e., the function obtained by actually doing the summation at each point)

3.2. **Inverses of each other?** The next natural question is whether the two mappings are inverses of each other. This breaks up into two questions:

(1) Start with an infinitely differentiable function f. Take its Taylor series and now consider the interval of convergence of this power series. Does the Taylor series converge to f on its interval of convergence?

(2) Start with a power series with a nonzero (positive or infinite) radius of convergence. Consider the function to which it converges on the interval of convergence, and take the Taylor series of that function. Do we get back the original power series?

The answer to the first question is *no* and the answer to the second question is *yes*.

3.3. **The answer to (2) is yes.** If we start with a series, look at the function it converges to, and take the Taylor series of that function, we get back the original series. This can easily be verified from the differentiability theorem.

The way this works is as follows: suppose f is the function obtained from the power series $\sum a_k x^k$. Then, by the differentiability theorem applied m times and evaluate at 0. The value at 0 turns out to be $k! a_k$. This means that $f^{(k)}(0) = k! a_k$. Rearranging, we see that $a_k = f^{(k)}(0)/k!$, so the power series we started with has the same coefficients as the Taylor series of f.

3.4. **Why the answer to (1) is no.** There are three things that could go wrong:

- The Taylor series for the function does not converge anywhere, so the question that (1) poses becomes meaningless.
- The Taylor series for the function converges, but not to the original function. An example of this is the function described below.
- The Taylor series does converge to the function, but the interval of convergence of the Taylor series is a lot smaller than the domain of definition of the function.

Consider the function:

$$f(x) := \{ \begin{array}{rl} e^{-1/x^2}, & x \neq 0 \\ 0, & x = 0 \end{array}$$

By definition $f(0) = 0$. We can check that f is continuous at 0, and in fact, it is infinitely differentiable at 0 and all its derivatives take the value 0 at 0. Thus, the Taylor series for f is the zero series. However, f is *not* the zero function.

On the other hand, for polynomials, the sine function, the cosine function, and the exponential function, the power series *does* converge back to the original function, and the radius of convergence is all of $\mathbb{R}$. One way to establish this (which we did in the previous lecture) is using the Lagrange formula for the remainder, and show that, as $n \to \infty$, the remainder goes to zero.

For some other functions, the power series converges to the function on a small interval, even though the function is defined on a bigger interval. For instance, the function $\ln(1+x)$ has a power series that converges on $(-1, 1]$ even though the function is defined on $(-1, \infty)$. Intuitively, the interval of convergence of the power series must be roughly symmetric about 0, whereas the domain of definition of the function may be a lot bigger.

We can make this more precise. For any infinitely differentiable function f whose domain D is an open interval containing 0 (possibly infinite in one direction) and such that the function cannot be extended immediately beyond D, then the radius of convergence of the Taylor series of f cannot be more than the *smaller* of the two sides about 0. For instance, for a function defined on $(-1, 3)$, the radius of convergence of the Taylor series about 0 cannot be more than 1, while for a function defined on $(-\infty, 2.3)$, the radius of convergence of the Taylor series cannot be more than 2.3.

Thus, we can say offhand that, for the function tan, the radius of convergence of its power series cannot be more than $\pi/2$, because the function goes off to ∞ at $\pi/2$ and to $-\infty$ at $-\pi/2$.

Is it possible for the Taylor series to converge to the function on an open interval about 0, but not on the largest possible symmetric interval? Indeed it is. An example of this is the arctan function. This function is defined and infinitely differentiable on all of $\mathbb{R}$. However, its interval of convergence is $[-1, 1]$. This is an application of the integration theorem, the theorem on convegence of alternating series, and Abel's theorem, and we will see this in the next section.

3.5. **Power series centered at points other than** 0. We discussed Taylor series centered at points a other than 0. We can analogously discuss power series centered at any point λ: there are series of the form $\sum a_k(x-\lambda)^k$. The results stated above are easy to translate (in both senses of the word) to this putatively more general context.

3.6. **Real-analytic functions: where the answer to (1) is yes.** A *globally analytic function* is a function that has a Taylor series that converges to the function everywhere. Globally analytic functions are everywhere infinitely differentiable, but infinitely differentiable functions need not be globally analytic (as the e^{-1/x^2} example illustrates). Here are some important facts about globally analytic functions:

(1) Sine, cosine, the exponential function, and polynomials are all globally analytic functions.
(2) Globally analytic functions are closed under addition, subtraction, and scalar multiplication. Moreover, addition of functions corresponds to addition of the series, subtraction of functions corresponds to subtraction of series, and scalar multiplication of a function corresponds to scalar multiplication of its series. Thus, the globally analytic functions form a $\mathbb{R}$-vector space.
(3) Globally analytic functions are closed under multiplication. Multiplication of functions corresponds to multiplication of series, where the multiplication of power series is carried out in a manner generalizing multiplication of polynomials. Combining with the previous observation, we obtain that globally analytic functions form a $\mathbb{R}$-algebra.
(4) Globally analytic functions are closed under differentiation and integration. Any derivative of a globally analytic function is globally analytic. Any antiderivative of a globally analytic function is globally analytic.
(5) Globally analytic functions are closed under composition. Composition of functions corresponds to basically plugging in place of the x in one series the entirety of the other series. For instance, for $\sin(x^2)$, we plug in x^2 in place of x in the power series of x. For $\sin(\cos x)$, we plug in the entire power series for $\cos x$ in place of x in the power series for sin.
(6) Globally analytic functions are *not* closed under taking quotients. For instance, $1/(x^2+1)$ is not globally analytic. This is a very important observation because it is one of the reasons why we can exit the world of globally analytic functions.

In addition to globally analytic functions, we are also interested in locally analytic functions: functions whose Taylor series have nonzero radius of convergence and converge to the function on that nonzero radius. Some examples of locally analytic functions that are not globally analytic include the tangent, arc tangent, and logarithm function. Analogues of all the observations (except the last one) about globally analytic functions can be made about locally analytic functions. Note that when we add two functions with different radii of convergence, the radius of convergence of the sum could be as low as the *minimum* of the two radii of convergence.

4. Playing around with power series, differentiation, and integration

4.1. **Rational functions: basics.** We begin by exploring the power series for rational functions. We already know, by the geometric series formula, that:

$$\frac{1}{1-x} = 1 + x + x^2 + \dots$$

with the radius of convergence equal to 1.

Some corollaries and related formulas, all with radius of convergence 1, are:

$$\begin{aligned}
\frac{1}{1+x} &= 1 - x + x^2 - x^3 + x^4 - \dots \\
\frac{1}{1+x^2} &= 1 - x^2 + x^4 - x^6 + x^8 - x^{10} + x^{12} - \dots \\
\frac{1}{1-x^n} &= 1 + x^n + x^{2n} + \dots \\
\frac{1}{1+x^n} &= 1 - x^n + x^{2n} - x^{3n} + x^{4n} - \dots \\
\frac{1}{(1-x)^2} &= 1 + 2x + 3x^2 + \dots
\end{aligned}$$

We can use these to quickly determine the power series expansions for $x/(1+x^2)$, $1/(1+x^2)^2$, and so on, using composition and multiplication.

4.2. **Logarithm function.** Using the integration theorem, we see that:

$$\int_0^x \frac{dt}{1+t} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots$$

The left side is $\ln(1+x)$ for $x \in (-1, \infty)$, and we thus get that, at least for $x \in (-1, 1)$:

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots$$

The limit of the function at -1 is $-\infty$, and the limit of the function at 1 is $\ln 2$. By the theorem on alternating series, the series converges at 1. Hence, by Abel's theorem, the series converges at 1 to $\ln 2$, and we get:

$$\ln 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \dots$$

which can be written more compactly as:

$$\ln 2 = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k}$$

4.3. **Arc tangent function.** Using the integration theorem, we see that:

$$\int_0^x \frac{dt}{1+t^2} = x - \frac{x^3}{3} + \frac{x^5}{5} - \dots$$

with the radius of convergence again equal to 1. We already know that the left side is $\arctan x$, so we obtain:

$$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots$$

The radius of convergence is 1, so the series converges absolutely on $(-1, 1)$. Does the series converge at -1 and at 1? It turns out that:

- By the theorem of alternating series, the series does converge at 1. Thus, by Abel's theorem, we obtain that $\arctan 1$ equals the value of the series at 1. We thus obtain:

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \dots$$

- The same theorem works at -1, giving:

$$\frac{-\pi}{4} = -1 + \frac{1}{3} - \frac{1}{5} + \frac{1}{7} - \dots$$

4.4. **Other rational functions.** Consider:

$$\frac{1}{(1-x)(2-x)}$$

Using Taylor series to find its expansion is tedious. Instead, we use the known expansion for $(1-x)^{-1}$ and $(1-(x/2))^{-1}$ and get:

$$\frac{1}{2}(1 + x + x^2 + \dots)(1 + (x/2) + (x/2)^2 + \dots)$$

The radii of convergence for the two series are 1 and 2 respectively, and so the overall radius of convergence is 1. We can now do polynomial-style multiplication to determine the coefficients of the power series.

4.5. **Integrating the unintegrable.** Earlier in the course, we encountered many functions that could not be integrated in terms of other elementary functions. Power series, however, allow us to integrate many more functions. Specifically, if we can express a function in terms of a power series, we can express its integral in terms of a power series, even if there is no description of the power series directly in terms of elementary functions.

We use this to re-explore some of the unintegrable functions seen earlier in the course.

Recall the function, which we'll here call ERF:

$$ERF(x) = \int_0^x e^{-t^2}\, dt$$

The power series expansion for e^{-x^2} is:

$$1 - \frac{x^2}{1!} + \frac{x^4}{2!} - \frac{x^6}{3!} + \dots$$

Integrating term wise, we obtain that:

$$ERF(x) = x - \frac{x^3}{1! \cdot 3} + \frac{x^5}{2! \cdot 5} - \frac{x^7}{3! \cdot 7} + \dots$$

This power series expansion allows us to calculate $ERF(x)$ to any desired degree of accuracy without having to use the various techniques of integration such as partitions, upper sums, and lower sums.

Another function that we looked at in the part was:

$$Si(x) = \int_0^x \frac{\sin t}{t}\, dt$$

Again, we can write:

$$\frac{\sin x}{x} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \dots$$

Integrating term wise, we obtain that:

$$Si(x) = x - \frac{x^3}{3 \cdot 3!} + \frac{x^4}{5 \cdot 5!} - \frac{x^6}{7 \cdot 7!} + \dots$$

5. Eyeballing to determine convergence

5.1. **Hierarchy of functions.** Here is a useful hierarchy to remember:

(1) Double exponential and other such monstrosity.
(2) Exponential in $x^r, r > 1$.
(3) Factorial, or Γ function. Roughly, exponential in $x \ln x$.
(4) Exponential or geometric.
(5) Exponential in $x^r, r < 1$.
(6) Exponential in $(\ln x)^r, r > 1$.
(7) Polynomial, or about $x^r, r > 0$.
(8) Polynomial in the logarithm.

This is a rough hierarchy of the important functions. Note that the ones above geometric are double exponential, exponential in $x^r, r > 1$, and factorial. These are the functions that totally dominate the behavior of geometric series. If this kind of function is in the numerators of terms of a power series, the power series has radius of convergence 0. If this kind of function is in the denominators of terms of power series, the power series has radius of convergence ∞.

Thus, the following power series have radius of convergence 0:

(1) $\sum k! x^k$
(2) $\sum 2^{k^2} x^k$
(3) $\sum 2^{2^k} x^k$

On the other hand, the following power series have radius of convergence ∞:

(1) $\sum x^k/k!$ or similar power series.
(2) $\sum e^{-k^2} x^k$
(3) $\sum e^{-2^k} x^k$.

On the middle hand, if the coefficients of the power series are exponential or subexponential, then the power series usually has a finite radius of convergence. We consider some of these cases below.

5.2. **Power series where the coefficients are exponential.** If the coefficients are exponential, then these determine the radius of convergence, For instance $\sum 2^k x^k$ has radius of convergence $1/2$.

5.3. **Power series where the coefficients are rational functions.** Consider a situation where the coefficients a_k are rational functions. In this case, the geometric behavior of the series dominates over the coefficients, and the radius of convergence is 1.

The rational function does affect something: convergence at the boundary, i.e., convergence at 1 and -1. Here, we follow two major rules:

(1) The rule for alternating series usually settles zero or one of the two boundary points, where the terms of the series have alternating signs.
(2) For the other boundary point(s), where all terms have the same sign, the criterion for convergence is that the degree of the denominator should be at least 2 more than the degree of the numerator.

Thus, $\sum x^k/k^2$ converges on $[-1,1]$, because the series has positive terms at both boundary points and the denominator has degree 2 more than the numerator. $\sum x^k/k$ converges on $[-1,1)$ – convergence at -1 because of alternating series, and diverging at 1 because the degree gap between the numerator and the denominator is just 1.

5.4. **Power series where the coefficients are product of rational function and exponential function.** In this case, the exponential part determines the radius of convergence, and the rational function part determines the issue of convergence at the boundary.

For instance, consider the power series:

$$\sum_{k=0}^{\infty} \frac{(2x)^k}{k^2+1}$$

The coefficients have an exponential component 2^k and a subexponential component $1/(k^2+1)$. The exponential component determines the radius of convergence, which in this case becomes $1/2$. The subexponential components determines whether the series converges at the endpoints $-1/2$ and $1/2$. In this case, since $\sum 1/(k^2+1)$ is absolutely convergent, convergence occurs at *both* endpoints.

Consider another power series:

$$\sum_{k=0}^{\infty} \frac{x^{2k}}{k\ln(2+k)3^k}$$

The radius of convergence is $\sqrt{3}$. At the endpoint $\sqrt{3}$, we get:

$$\sum_{k=0}^{\infty} \frac{1}{k\ln(2+k)}$$

which diverges by the integral test. At the other endpoint $-\sqrt{3}$, we get the same summation, which again diverges by the integral test. So the interval of convergence is $(-\sqrt{3},\sqrt{3})$.