LIMIT COMPUTATION: FORMULAS AND TECHNIQUES, INCLUDING L'HÔPITAL'S RULE

MATH 153, SECTION 55 (VIPUL NAIK)

Corresponding material in the book: Sections 11.4, 11.5, 11.6.

What students should already know: Basic limit computations.

What students should definitely get: A good jolt that brings back into active memory all the past knowledge of limits. The statement of L'Hôpital's rule for 0/0 form and ∞/∞ form, how to apply it, and the important constraints and caveats.

What students should hopefully get: The notion of order of a zero, how the LH rule can be effectively combined with stripping and substitution, and the intuition for when a limit is going to be zero, infinity, or finite and nonzero.

EXECUTIVE SUMMARY

Words ...

(1) L'Hôpital's rule for 0/0 form: Consider $\lim_{x \to c} f(x)/g(x)$ where $\lim_{x \to c} f(x) = \lim_{x \to c} g(x) = 0$. If both f and g are differentiable around c, then this limit is equal to $\lim_{x \to c} f'(x)/g'(x)$.

(2) Analogous statements hold for one-sided limits and in the case $c = \pm\infty$.

(3) The 0/0 form LH rule *cannot* be applied if the numerator does not limit to zero or if the denominator does not limit to zero.

(4) L'Hôpital's rule for ∞/∞ form: Consider $\lim_{x \to c} f(x)/g(x)$ where $\lim_{x \to c} f(x) = \pm\infty$ and $\lim_{x \to c} g(x) = \pm\infty$. If both f and g are differentiable around c, then this limit is equal to $\lim_{x \to c} f'(x)/g'(x)$.

(5) Analogous statements hold for one-sided limits and for $c = \pm\infty$.

(6) For a function f having a zero at c, the *order* of the zero at c is the least upper bound of β such that $\lim_{x \to c} |f(x)|/|(x - c)|^\beta = 0$. At this least upper bound, the limit is usually finite and nonzero. For larger values of β, the limit is undefined.

(7) Given a quotient f/g for which we need to calculate the limit at c, the limit is zero if the order of the zero for f is greater than the order for g, and undefined if the order of the zero for f is less than the order for g. When the orders are the same, the limit could potentially be a finite nonzero number.

Actions ...

(1) For polynomial functions and other continuous functions, we can calculate the limit at a point by evaluating at the point. For rational functions, we can cancel common factors between the numerator and the denominator till one of them becomes nonzero at the point.

(2) There is a bunch of basic limits that translate to saying that for the following functions f: $f(0) = 0$ and $f'(0) = 1$, which is equivalent to saying that $\lim_{x \to 0} f(x)/x = 1$. These functions are $\sin x$, $x \mapsto \ln(1 + x)$, $x \mapsto e^x - 1$, $x \mapsto \tan x$, $x \mapsto \arcsin x$, and $x \mapsto \arctan x$.

(3) For all the functions f of the above kind (that we call *strippable*), the following is true: in any multiplicative situation, if the input to the function goes to zero in the limit, the function can be stripped off.

(4) Two other basic limits are $\lim_{x \to 0} (1 - \cos x)/x^2 = 1/2$ and $\lim_{x \to 0} (x - \sin x)/x^3 = 1/6$. These can be obtained using the LH rule.

(5) Typically, to compute limits, we can combine the LH rule, stripping, and removing multiplicative components that we can calculate directly.

(6) Applying the LH rule 0/0 form pushes the orders of both the numerator and the denominator down by one.

(7) It is also useful to remember that logarithmic functions are dominated by polynomial functions, which in turn are dominated by exponential functions. These facts can be seen in various ways, including the LH rule.

1. Limits: some of the key ideas

1.1. The basic process for real limits at finite values.

(1) For continuous functions, evaluate at the point.
(2) If evaluation gives a form that is determinately undefined (such as a quotient where the numerator approaches a finite nonzero number and the denominator approaches zero) then declare that the limit does not exist as a finite number. We may still be interested in whether the limit exists as $+\infty$ or $-\infty$, and we can follow the rules to find out if that is the case.
(3) If evaluation gives an *indeterminate form*, manipulate the expression to get an *equivalent function* that can be evaluated at the point. For quotients, the process typically involves the cancellation of some common factors between the numerator and the denominator.

1.2. Some important basic cases.

(1) For the limit of a rational function $p(x)/q(x)$ at $x = a$, try evaluating at a. If evaluation gives a $0/0$ form, that means that $x - a$ divides both $p(x)$ and $q(x)$. Factorize and cancel this common factor. Now evaluate again. If we still get a $0/0$ form, there is another $x - a$ common to both the numerator and the denominator, and it can be canceled again. The process continues till either the numerator or the denominator has no factor of $x - a$. The limit is (a) finite and nonzero if $x - a$ divides neither numerator nor denominator, (b) zero if $x - a$ divides the numerator but not the denominator, and (c) undefined if $x - a$ divides the denominator but not the numerator.
(2) For limits involving trigonometric functions, we can use trigonometric identities in case the expression is of a $0/0$ form. Remember to *first evaluate and check*.
(3) For limits involving a mix of trigonometric functions and polynomial functions, we use the standard limit $\lim_{x \to 0}(\sin x)/x = 1$ and its many corollaries, such as $\lim_{x \to 0}(\tan x)/x = 1$ and $\lim_{x \to 0}(1 - \cos x)/x^2 = 1/2$.

1.3. Some key important limits.
Some key nontrivial limits that we need to remember are:

$$\lim_{x \to 0} \frac{\sin x}{x} = 1$$
$$\lim_{x \to 0} \frac{1 - \cos x}{x^2} = \frac{1}{2}$$
$$\lim_{x \to 0} \frac{\tan x}{x} = 1$$
$$\lim_{x \to 0} \frac{e^x - 1}{x} = 1$$
$$\lim_{x \to 0} \frac{\ln(1 + x)}{x} = 1$$
$$\lim_{x \to 0} \frac{\arcsin x}{x} = 1$$
$$\lim_{x \to 0} \frac{\arctan x}{x} = 1$$

Each of these limits has an interpretation as a derivative: it simply says that the numerator functions all have value 0 and derivative 1 at $x = 0$. We will see this in more detail later.

1.4. Seeing these limits in composites.
Consider the limit:

$$\lim_{x \to 0} \frac{\ln(1 + \sin x)}{x}$$

Note that the numerator is obtained by *composing* the sin function and the $t \mapsto \ln(1 + t)$ function. This composition of functions translates, in computational terms, to a *product* of limits. In a process similar to the chain rule, we write the limit as:

$$\lim_{x \to 0} \frac{\ln(1 + \sin x)}{\sin x} \frac{\sin x}{x}$$

We now split the limit:

$$\lim_{x \to 0} \frac{\ln(1 + \sin x)}{\sin x} \lim_{x \to 0} \frac{\sin x}{x}$$

Now, the second limit is 1. The first limit is *also* 1, because as $x \to 0$, we also have $\sin x \to 0$, and we can thus apply the limit formula.

This compositional breaking up is sometimes accompanied by a few constants here and there. For instance:

$$\lim_{x \to 0} \frac{\sin(2 \ln(1 + x^2))}{e^{x^2} - 1}$$

This is too tedious to write in full, but the numerator is, in the limit, multiplicatively the same as $2 \ln(1 + x^2)$ (i.e., we can strip the sin off, which is twice of $\ln(1 + x^2)$, which is multiplicatively the same as x^2. The denominator is multiplicatively the same as x^2. So, the overall quotient is 2.

Similarly, consider:

$$\lim_{x \to 0} \frac{\sin(ax)}{\sin(bx)}$$

where $a, b \in \mathbb{R}$ and $b \neq 0$. In this case, the numerator is multiplicatively the same as ax, and the denominator is multiplicatively the same as bx, so the quotient is a/b. On the other hand:

$$\lim_{x \to 0} \frac{\cos(ax)}{\cos(bx)}$$

succumbs to a direct evaluation: at 0, both the numerator and the denominator equal 1, so the quotient equals 1.

1.5. **Heavier and lighter denominators.** Consider the limit:

$$\lim_{x \to 0} \frac{\sin x}{x^2}$$

Here, the numerator is roughly like x (multiplicatively), so the limit becomes:

$$\lim_{x \to 0} \frac{x}{x^2} = \lim_{x \to 0} \frac{1}{x}$$

which is undefined. On the other hand:

$$\lim_{x \to 0} \frac{\sin x}{\sqrt{x}} = \lim_{x \to 0} \frac{x}{\sqrt{x}} = \lim_{x \to 0} \sqrt{x} = 0$$

Basically, we can *strip off* things which multiplicatively go to 1 and apply the usual rules of limits for rational functions to whatever is left (more on this in the next section).

1.6. **Sum and product breakups.** Also consider:

$$\lim_{x \to 0} \frac{\sin(\pi x) \sin(ex)}{x^2}$$

You can convince yourself that the limit in this case is πe. This essentially arises by breaking up the limit as the product of two limits. Similarly:

$$\lim_{x \to 0} \frac{\sin(\sqrt{2}x) + \tan(\sqrt{3}x)}{x}$$

is $\sqrt{2} + \sqrt{3}$ – again obtained by breaking up.

2. Limits at infinity

We have previously reviewed the limits of rational functions at infinity, so we will not cover them again. Rather, we concentrate on a general approach here: to calculate $\lim_{x \to \infty} f(x)$, put $t = 1/x$ and calculate $\lim_{t \to 0^+} f(1/t)$.

2.1. The original identities transformed to limits at infinity.

$$\lim_{x \to \infty} x \sin(1/x) = 1$$

$$\lim_{x \to \infty} x^2 (1 - \cos(1/x)) = \frac{1}{2}$$

$$\lim_{x \to \infty} x \tan(1/x) = 1$$

$$\lim_{x \to \infty} x(e^{1/x} - 1) = 1$$

$$\lim_{x \to \infty} x \ln(1 + (1/x)) = 1$$

The last identity is particularly important, and is also often written in exponential form:

$$\lim_{x \to \infty} \left[1 + \frac{1}{x}\right]^x = e$$

In general, whenever we have a limit of the form:

$$\lim_{x \to \infty} \left[1 + \frac{1}{f(x)}\right]^{g(x)}$$

where, as $x \to \infty$, we also have $f(x) \to \infty$, then this just becomes:

$$\lim_{x \to \infty} \exp \left[\frac{g(x)}{f(x)}\right]$$

Try to convince yourself of this.

3. Stripping in full glory

This is an intuitive way of thinking about and calculating *some* limits that incorporates the principles of chaining discussed above, without rewriting all the intermediate steps in gory details. We had a sneak peek of this approach in Math 152, but now we go the whole hog.

3.1. Simple examples. To motivate stripping, let us look at a fancy example:

$$\lim_{x \to 0} \frac{\sin(\tan(\sin x))}{x}$$

This is a composite of three functions, so if we want to chain it, we will chain it as follows:

$$\lim_{x \to 0} \frac{\sin(\tan(\sin x))}{\tan(\sin x)} \frac{\tan(\sin x)}{\sin x} \frac{\sin x}{x}$$

We now split the limit as a product, and we get:

$$\lim_{x \to 0} \frac{\sin(\tan(\sin x))}{\tan(\sin x)} \lim_{x \to 0} \frac{\tan(\sin x)}{\sin x} \lim_{x \to 0} \frac{\sin x}{x}$$

Now, we argue that each of the inner limits is 1. The final limit is clearly 1. The middle limit is 1 because the inner function $\sin x$ goes to 0. The left most limit is 1 because the inner function $\tan(\sin x)$ goes to 0. Thus, the product is $1 \times 1 \times 1$ which is 1.

If you are convinced, you can further convince yourself that the same principle applies to a much more convoluted composite:

$$\lim_{x \to 0} \frac{\sin(\sin(\tan(\sin(\tan(\tan x)))))}{x}$$

However, *writing that thing out takes loads of time.* Wouldn't it be nice if we could just strip off those sins and tans? In fact, we can do that.

The key stripping rule is this: *in a multiplicative situation* (i.e. there is no addition or subtraction happening), if we see something like $\sin(f(x))$ or $\tan(f(x))$, and $f(x) \to 0$ in the relevant limit, then we can strip off the sin or tan. In this sense, both sin and tan are *strippable* functions. A function g is strippable if $\lim_{x\to 0} g(x)/x = 1$.

The reason we can strip off the sin from $\sin(f(x))$ is that we can multiply and divide by $f(x)$, just as we did in the above examples.

Stripping can be viewed as a special case of the l'Hopital rule as well, but it's a much quicker shortcut in the cases where it works.

Thus, in the above examples, we could just have stripped off the sins and tans all the way through.

Here's another example:

$$\lim_{x\to 0} \frac{\sin(2\tan(3x))}{x}$$

As $x \to 0$, $3x \to 0$, so $2\tan 3x \to 0$. Thus, we can strip off the outer sin. We can then strip off the inner tan as well, since its input $3x$ goes to 0. We are thus left with:

$$\lim_{x\to 0} \frac{2(3x)}{x}$$

Cancel the x and get a 6. We could also do this problem by chaining or the l'Hopital rule, but stripping is quicker and perhaps more intuitive.

Here's yet another example:

$$\lim_{x\to 0} \frac{\sin(x\sin(\sin x))}{x^2}$$

As $x \to 0$, $x\sin(\sin x) \to 0$, so we can strip off the outermost sin and get:

$$\lim_{x\to 0} \frac{x\sin(\sin x)}{x^2}$$

We cancel a factor of x and get:

$$\lim_{x\to 0} \frac{\sin(\sin x)}{x}$$

Two quick sin strips and we get x/x, which becomes 1.

Yet another example:

$$\lim_{x\to 0} \frac{\sin(ax)\tan(bx)}{x}$$

where a and b are constants. Since this is a multiplicative situation, and $ax \to 0$ and $bx \to 0$, we can strip the sin and tan, and get:

$$\lim_{x\to 0} \frac{(ax)(bx)}{x}$$

This limit becomes 0, because there is a x^2 in the numerator and a x in the denominator, and cancellation of one factor still leaves a x in the numerator.

Here is yet another example:

$$\lim_{x\to 0} \frac{\sin^2(ax)}{\sin^2(bx)}$$

where a, b are nonzero constants. We can pull the square out of the whole expression, strip the sins in both numerator and denominator, and end up with a^2/b^2.

Here's another example:

$$\lim_{x \to 0} \frac{\arcsin(2\sin^2 x)}{x \arctan x}$$

Repeated stripping reveals that the answer is 2. Note that arcsin and arctan are also strippable because $\lim_{x\to 0}(\arcsin x)/x = 1$ and $\lim_{x\to 0}(\arctan x)/x = 1$.

3.2. **A more complicated example.** Here's another example:

$$\lim_{x \to 0} \frac{\ln(1 + \sin(3x \sin(2\sin x)))}{x \tan(e^x - 1)}$$

We note that as $x \to 0$, we have $3x \sin(2 \sin x) \to 0$, so sin of that goes to 0, so we can strip off $\ln(1 + \_)$ and get:

$$\lim_{x \to 0} \frac{\sin(3x \sin(2\sin x))}{x \tan(e^x - 1)}$$

We can now strip off the outer sin, then take 3 outside of the limit, to get:

$$3 \lim_{x \to 0} \frac{x \sin(2 \sin x)}{x \tan(e^x - 1)}$$

The x and x cancel, and we get:

$$3 \lim_{x \to 0} \frac{\sin(2 \sin x)}{\tan(e^x - 1)}$$

Now, the sin gets stripped off from the numerator, and the tan from the denominator, because the inputs to both functions go to 0. So we are left with:

$$6 \lim_{x \to 0} \frac{\sin x}{e^x - 1}$$

Finally, the sin and the $\exp -1$ can be stripped off and we just get the answer as 6.

Note that sin, tan, $\ln(1 + \_)$, and $\exp(\_) - 1$ are all strippable functions, because for each of these, the limit of the function divided by its input goes to 1 as the input goes to 0.

3.3. **Strip with discretion.** Consider:

$$\lim_{x \to 0} \frac{\sin(x \sin(\pi \cos x))}{x^2}$$

Here, the outer sin can be stripped off, because its input goes to 0, so we get:

$$\lim_{x \to 0} \frac{x \sin(\pi \cos x)}{x^2}$$

Cancel the x and we get:

$$\lim_{x \to 0} \frac{\sin(\pi \cos x)}{x}$$

However, it is *not* legitimate now to strip off the sin at this stage, because the input to the sign, namely $\pi \cos x$, does not tend to 0 as $x \to 0$. There are other right ways of doing this problem, which we will discuss later, but stripping is not one of them.

3.4. When you can't strip. The kind of situations where we are not allowed to strip are where the expression $\sin(f(x))$ is not just multiplied but is being added to or subtracted from something else. For instance, in order to calculate the limit:

$$\lim_{x \to 0} \frac{x - \sin x}{x^3}$$

We are *not* allowed to strip.

Roughly speaking, what goes wrong if we just strip is something like this: x and $\sin x$ are *multiplicatively* the same, but they are *additively* different, near 0. When taking the difference, we are picking out on a very thin, very small additive difference that is invisible multiplicatively.

What about $1 - \cos(f(x))$? If we see something like $1 - \cos(f(x))$ with $f(x) \to 0$ in the limit, then we know that it "looks like" $f(x)^2/2$ and we can replace it by $f(x)^2/2$. This can be thought of as a sophisticated version of stripping. For instance:

$$\lim_{x \to 0} \frac{(1 - \cos(5x^2))}{x^4}$$

Here $f(x) = 5x^2$, so the numerator is like $(5x^2)^2/2$, and the limit just becomes $25/2$.

4. Understanding indeterminate form 0/0, towards L'Hôpital's rule

The expression $0/0$ is undefined. It does not make sense to divide 0 by 0, because any real number times 0 is 0. However, $(\to 0)/(\to 0)$ is an *indeterminate form*. In other words, if we're trying to compute the limit:

$$\lim_{x \to c} \frac{f(x)}{g(x)}$$

such that $\lim_{x \to c} f(x) = 0$ and $\lim_{x \to c} g(x) = 0$, then we cannot *prima facie* say anything for sure about the limit of the quotient. Let's try to understand the source of the uncertainty a little more carefully.

4.1. The rate of approach to zero. The key thing to understand is that even though two quantities may approach 0, they may approach it at very different rates. For instance, consider $f(x) = 4x$ and $g(x) = 7x$. As x approaches 0, both $f(x)$ and $g(x)$ approach 0. However, for small x, $f(x)$ comes a little closer to 0 than $g(x)$. The rate of approach for f is 4 and the rate of approach of g is 7. The quotient thus approaches $4/7$.

Essentially, if $\lim_{x \to c} f(x) = 0$, then the derivative $f'(c)$ describes the rate at which f approaches 0 at c. The smaller the value of $f'(c)$, the faster f approaches 0. Thus, the derivative measures the rate of approach to zero. To determine the limit of the quotient of two functions, we need to compare their rates of approach to zero. This is the intuitive basis for L'Hôpital's rule.

4.2. A primitive form of the rule. This primitive form says that if f and g are continuous at c and $f(c) = g(c) = 0$, and if $f'(c)$ and $g'(c)$ are both defined, with $g'(c) \neq 0$, then we have:

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{f'(c)}{g'(c)}$$

Here is how we can think of this intuitively. Near the point $(c, 0)$, the graph of f looks very close to the straight line passing through $(c, 0)$ with slope $f'(c)$ and the graph of g looks very close to the straight line passing through $(c, 0)$ with slope $g'(c)$. The quotient of the function values is, in the limit, the same as the quotient of the slopes of these lines.

4.3. A more sophisticated form of the rule. The primitive version above works if g has nonzero derivative at c. A slight modification of it works if g has zero derivative at c. Namely, if $\lim_{x \to c} f(x) = \lim_{x \to c} g(x) = 0$ and f and g are both differentiable around c with g not identically zero around c, then:

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} \frac{f'(x)}{g'(x)}$$

The advantage of this is that this works even if $g'(c)$ is zero or undefined, i.e., it works even at critical points of c. After one application of this, there are three possibilities:

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(1) The new limit can be directly computed because the denominator approaches a nonzero number.
(2) The denominator approaches zero and the numerator approaches a nonzero number. The limit in this case is undefined.
(3) Both the numerator and denominator approach 0. We thus *again* have the indeterminate form $(\to 0)/(\to 0)$. We can now use the usual suite of techniques, which includes applying L'Hôpital's rule yet again.

There are actually more possibilities, because the numerator and/or denominator may approach ∞. But we ignore those possibilities for now just to keep the narrative simple.

4.4. **Caveat for applying the rule.** The rule as stated here should be applied *only* when both the numerator and the denominator approach 0. (We shall later see that it also applies when both approach ∞). This is important because applying the rule in other situations yields incorrect answers. Simply speaking, it makes sense to compare the *rates of approach to zero* only when they both *are actually heading to zero*.

4.5. **A formal proof of the rule.** We prove here the primitive version of the rule, which assumes that f and g are continuous and differentiable at c with $f(c) = g(c) = 0$. We consider:

$$\lim_{x \to c} \frac{f(x)}{g(x)}$$

Using $f(c) = g(c) = 0$, we can rewrite this as:

$$\lim_{x \to c} \frac{f(x) - f(c)}{g(x) - g(c)}$$

Dividing both numerator and denominator by $x - c$, we obtain:

$$\lim_{x \to c} \frac{\frac{f(x) - f(c)}{x - c}}{\frac{g(x) - g(c)}{x - c}}$$

The limit of the quotients is the quotient of the limits, and we obtain:

$$\frac{f'(c)}{g'(c)}$$

The same idea can be extended to the more general case, but the details get more messy so we avoid this.

4.6. **The rule applies to one-sided limits and limits for** $c = \infty$. L'Hôpital's rule also applies for one-sided limits (i.e., the left hand limit and the right hand limit) for finite values of c and for the limit at $c = \infty$ or at $c = -\infty$. When applying to one-sided limits, we take the corresponding one-sided limit of the derivative. Thus, if $\lim_{x \to c^-} f(x) = \lim_{x \to c^-} g(x) = 0$, and if f and g are differentiable on the immediate left of c, we have:

$$\lim_{x \to c^-} \frac{f(x)}{g(x)} = \lim_{x \to c^-} \frac{f'(x)}{g'(x)}$$

Similarly, if $\lim_{x \to \infty} f(x) = \lim_{x \to \infty} g(x) = 0$, and if f and g are differentiable for large inputs, we have:

$$\lim_{x \to \infty} \frac{f(x)}{g(x)} = \lim_{x \to \infty} \frac{f'(x)}{g'(x)}$$

5. Applying L'Hôpital's rule

5.1. **Simple applications.** Consider the limit:

$$\lim_{x \to 0^+} \frac{\sin x}{\sqrt{x}}$$

This is a 0/0 form. Our earlier way of doing this limit was simply to strip off the sin in front of the x, notice that we get $x/\sqrt{x} = \sqrt{x}$, which goes to 0. Now, we try the LH rule, and we get:

$$\lim_{x \to 0^+} \frac{\cos x}{1/(2\sqrt{x})}$$

A good shorthand for the application of this rule, used in the book, is to write it using $\overset{*}{=}$, as follows:

$$\lim_{x \to 0^+} \frac{\sin x}{\sqrt{x}} \overset{*}{=} \lim_{x \to 0^+} \frac{\cos x}{1/(2\sqrt{x})} = \lim_{x \to 0^+} 2\sqrt{x}\cos x$$

We now evaluate and obtain 0.

Notice that in the earlier procedure (that did not use the LH rule), we did not obtain any factor of 2 – we just obtained the limit of $\cos x$. In both cases, the final answer is 0. It turns out that with the LH rule, when the answer is 0 or ∞, using the LH rule often gives a lot of superfluous constants that ultimately don't matter, as compared to the primitive stripping techniques. On the other hand, in those cases where the limit is nonzero, all those constants *do* end up mattering.

5.2. **Another example.** Consider the example:

$$\lim_{x \to 0} \frac{x - \sin x}{x^3}$$

The first thing we try is evaluate. We notice that the numerator and denominator both approach zero. We also notice something else: there is no simple way to get started on the problem. For instance, if we try to split across the $-$ sign, we get $1/x^2 - (\sin x)/x^2$, both of which are undefined in the limit. Also, since the numerator is an additive mix of the algebraic and the trigonometric, we cannot directly use trigonometric identities.

So we use the LH rule. We obtain:

$$\lim_{x \to 0} \frac{x - \sin x}{x^3} \overset{*}{=} \lim_{x \to 0} \frac{1 - \cos x}{3x^2}$$

At this stage, we have two choices. The limit $\lim_{x \to 0}(1 - \cos x)/x^2$ is a standard trigonometric limit that we know to be $1/2$, and we can use this to get the answer of $1/6$. If, however, we have forgotten the answer and the procedure for computing this limit, we can use the LH rule again, noting that it is again the $(\to 0)/(\to 0)$ form. We get:

$$\lim_{x \to 0} \frac{x - \sin x}{x^3} \overset{*}{=} \lim_{x \to 0} \frac{1 - \cos x}{3x^2} = \lim_{x \to 0} \frac{\sin x}{6x}$$

Again, we can choose from two alternatives: use the well-known fact that $\lim_{x \to 0}(\sin x)/x = 1$, or conveniently forget it and hence *again* use the LH rule. We get:

$$\lim_{x \to 0} \frac{x - \sin x}{x^3} \overset{*}{=} \lim_{x \to 0} \frac{1 - \cos x}{3x^2} = \lim_{x \to 0} \frac{\sin x}{6x} = \lim_{x \to 0} \frac{\cos x}{6}$$

Now, we no longer have a $0/0$ form, so we *must* evaluate, and we obtain the answer $1/6$.

The point here is that the LH rule can be used in conjunction with other rules. If using the LH rule, you can choose to keep differentiating till you get something that is not of the $0/0$ form. But you can also choose to fork at any stage where you know how to compute the limit by other means.

5.3. **Combining the LH rule with stripping.** Stripping is the procedure where we throw off unnecessarily confusing layers. For instance, if the previous question had been:

$$\lim_{x \to 0} \frac{\sin(x - \sin x)}{\ln(1 + x^3)}$$

We could *strip off* the sin in the numerator and the $\ln(1 + \_)$ in the denominator to get:

$$\lim_{x \to 0} \frac{x - \sin x}{x^3}$$

Now, *this* limit can be evaluated using the LH rule. Note that even the original limit can be evaluated using the LH rule, but the derivatives get extremely messy, and these messy derivatives don't contribute anything to the discussion.

5.4. **Combining the LH rule with substitution.** Consider:

$$\lim_{x \to 0} \frac{x - \arcsin x}{x^3}$$

We can do this using the LH rule, but the derivatives of arcsin tend to get messy. It is smarter here to put $\theta = \arcsin x$, and obtain:

$$\lim_{\theta \to 0} \frac{\sin \theta - \theta}{\sin^3 \theta}$$

Using stripping, this can be transformed to:

$$\lim_{\theta \to 0} \frac{\sin \theta - \theta}{\theta^3}$$

which, as we have seen above, is $-1/6$.

5.5. **The order of a function near a zero.** The LH rule and prior discussions give rise to a notion of the order of a function near a zero of the function. Suppose that, as $x \to c$, we have $f(x) \to 0$. The *order* of f at c is the least upper bound of the set of β for which $\lim_{x \to c} f(x)/(x - c)^\beta = 0$.

For instance, for the function x at $c = 0$, the order is 1, because $\lim_{x \to 0} x/x^\beta = 0$ for all $\beta < 1$, whereas $\lim_{x \to 0} x/x^\beta$ is undefined for all $\beta > 1$. The number 1 is a threshold value at which the limit goes from being 0 to being undefined.

Here now are some observations:

(1) For any function f with the property that $f(c) = 0$ and $f'(c)$ is finite and nonzero, f has order 1 at c. In other words, $\lim_{x \to c} f(x)/(x - c)^\beta = 0$ for $\beta < 1$ and $\lim_{x \to c} f(x)/(x - c)^\beta$ is undefined for $\beta > 1$. At $\beta = 1$, the limit is $f'(c)$, which is a finite and nonzero number.

Examples for $c = 0$ are the functions $\sin x$, $\ln(1 + x)$, $\tan x$, $e^x - 1$, and $\arcsin x$.

(2) More generally, if $f(c) = 0$ and the smallest order of derivative of f that is nonzero at c is k (i.e., the kth derivative of f is nonzero at c and all lower order derivatives are zero), then the order of f at c is k. In other words, $\lim_{x \to c} f(x)/(x - c)^\beta = 0$ for all $\beta < k$ and $\lim_{x \to c} f(x)/(x - c)^\beta$ is undefined for all $\beta > k$. At $\beta = k$, the limit is finite and nonzero, and is in fact equal to $f^{(k)}(c)/k!$, as we can readily see by a repeated application of the LH rule.

Here are some examples of functions that have order 2 at $c = 0$: x^2, $1 - \cos x$, $e^x - 1 - x$, $\ln(1 + x^2)$.

Here are some examples of functions that have order 3 at $c = 0$: $x - \sin x$, $x - \tan x$, x^3.

(3) The function $f(x) := x^r$ for some $r > 0$ has order r at 0. r need not be an integer.

(4) Multiplication adds orders and function composition multiplies orders.

We can now interpret the LH rule as follows:

(1) Each application of the LH rule pushes down the order of both the numerator and the denominator by 1.

(2) When the numerator has higher order than the denominator, then the quotient approaches zero. In the case where both orders are positive integers, repeated application of the LH rule will get us to a situation where the denominator becomes nonzero (because the order of the zero in the denominator becomes zero) while the numerator is still zero (because the order of the zero in the denominator is still positive) – yes, you read that correct.

(3) When the numerator and the denominator have the same order, the quotient *could* approach something finite and nonzero. In most cases, repeated application of the LH rule gets us down to a quotient of two nonzero quantities.

(4) When the denominator has the higher order, the quotient has an undefined limit (the one-sided limits are usually $\pm\infty$). In the case where both orders are positive integers, repeated application of the LH rule will get us to a situation where the numerator becomes nonzero while the denominator is still zero (because the order of the zero is still positive).

It should be possible to determine these limits at 0 simply by inspection: $(\cos x)/x$, $(\sin x)/x^2$, $(\sin^2 x)/x$, $(1 - \cos x)/x^3$, $(x - \sin x)/x^2$.

One way of thinking of this is in terms of a race for domination. The order roughly measures the extent to which something is zero.

6. The LH rule for ∞/∞

This version of the LH rule states that if both the numerator and the denominator of a fraction go to $\pm\infty$ (with the two infinities not necessarily being the same) then the limit of the quotient equals the limit of the quotient of the derivatives. In symbols, if $f(x) \to \pm\infty$ and $g(x) \to \pm\infty$ as $x \to c$ and f and g are both differentiable around c with g not identically zero near c, we have:

$$\lim_{x\to c} \frac{f(x)}{g(x)} = \lim_{x\to c} \frac{f'(x)}{g'(x)}$$

This version of the LH rule should *not* be mixed with the earlier version. In particular, if the numerator approaches 0 and the denominator approaches $+\infty$, the LH rule does not apply. Similarly, if the numerator approaches $-\infty$ and the denominator approaches 0, the LH rule does not apply. (In both these cases, the limits are directly obvious: in the first case, the limit is 0, and in the second, the limit is undefined).

6.1. **Example of $e^x/p(x)$.** e^x races out polynomial functions. Specifically, for any nonzero polynomial function $p(x)$, $\lim_{x\to\infty} \frac{e^x}{p(x)} = \pm\infty$, with the sign of infinity depending upon the leading coefficient of p. The LH rule gives an easy proof. For a polynomial of degree k, applying the LH rule k times gives $\frac{e^x}{\text{constant}}$ where the constant in the denominator has the same sign as the leading coefficient of p (in fact, it is $k!$ times the leading coefficient of p). Since $e^x \to \infty$ as $x \to \infty$, we see tha tthis limit is ∞.

6.2. **Obtaining these from other indeterminate forms.** Suppose we have an indeterminate form other than $0/0$ or ∞/∞. We cannot directly use the LH rule. However, we can often do some manipulation to bring it into the $0/0$ or ∞/∞ form where the LH rule can be applied. In particular:

(1) A form of $(\to 0)(\to \infty)$ can be rewritten as $(\to 0)/(\to 0)$ or as $(\to \infty)/(\to \infty)$, by taking the reciprocal of one of the factors. For instance, $\lim_{x\to\infty} xe^{-x}$ can be written as $\lim_{x\to\infty} x/e^x$.
(2) A form of $\to \infty - \to \infty$ can often be converted to a $\to 0/ \to 0$ form. The trick is usually to take a common denominator and simplify. For instance, $(1/x) - (1/\sin x)$ can be rewritten as $(\sin x - x)/x^2$.