INTEGRATING RADICALS

MATH 153, SECTION 55 (VIPUL NAIK)

Corresponding material in the book: Section 8.4.

What students should already know: The definitions of inverse trigonometric functions. The differentiation and integration formulas for these. The differentiation formulas for the straight-up trigonometric functions.

What students should definitely get: The three key integral formulations: a^2-x^2, a^2+x^2, and x^2-a^2, and the mechanics of the trigonometric substitution for each. The procedure for completing the square term in quadratic functions and using this to integrate functions which have quadratics in denominators or with half-integer powers.

What students should hopefully get: The interpretation in terms of homogeneous degree, the contours of the relationship with inverse hyperbolic trigonometry.

Executive summary

Words ...

(1) Expressions of the form $a^2 + x^2$ (with $a > 0$) in the denominator or under the radical sign suggest the substitution $\theta = \arctan(x/a)$. With this substitution, $x = a\tan\theta$, $dx = a\sec^2\theta\,d\theta$, $a^2 + x^2 = a^2\sec^2\theta$, and $\sqrt{a^2+x^2} = a\sec\theta$. In the end, when substituting back, we use $\theta = \arctan(x/a)$, $\tan\theta = x/a$, $\sec\theta = \sqrt{a^2+x^2}/a$, $\cos\theta = a/\sqrt{a^2+x^2}$, and $\sin\theta = x/\sqrt{a^2+x^2}$. The first sentence of substitutions is useful when converting the given integrand into a trigonometric integrand. The second sentence is useful when converting the integrated answer back at the end. (This latter step is unnecessary when we are dealing with a definite integral and we transform limits simultaneously).

(2) For $a^2 - x^2$ under a squareroot, we have a similar substitution $\theta = \arcsin(x/a)$. For $x^2 - a^2$, we take $\theta = \arccos(a/x)$. It is useful to work out the forward and backward substitutions for these. (See the notes for the details of these substitutions). *It is strongly suggested that you internalize both the forward and the backward substitutions to the point where they become automatic. Memorization helps, but you should also be able to re-derive things on the spot as the need arises.*

(3) There is a little subtlety in these substitutions. When we take θ as arcsin, we know that $\cos\theta$ is nonnegative. Hence, when we simplify $\sqrt{a^2-x^2}$, we get $\sqrt{a^2\cos^2\theta}$. Because by assumption a is positive, and because $\cos\theta$ is nonnegative, we can write the answer as $a\cos\theta$. In other words, we know how exactly we can lift off the squareroot. Something similar happens when we are dealing with the tangent and secant functions: secant is nonnegative on the range of arc tangent. Unfortunately, tangent is *not* nonnegative on the entire range of arc secant, so we need to actually look at the region where we are carrying out the integration. In case both the upper and lower bounds of integration are greater than a, we know that we will in fact get $\tan\theta$.

Note: Some of you may find it useful to draw right triangles, as suggested in the book, if reading trigonometric ratios off triangles is easier for you than algebraic manipulation of trigonometric expressions.

Actions ...

(1) Trigonometric substitutions allow us to integrate things like $x^m(a^2+x^2)^{n/2}$. However, some special cases of these can be integrated without resort to trigonometric substitutions. For instance, when n is a nonnegative even integer, this is a sum of powers of x and can be integrated term wise. Also, if m is odd, we can do a u-substitution with $u = a^2 + x^2$.

(2) Similar remarks apply to expressions involving $\sqrt{a^2-x^2}$ and $\sqrt{x^2-a^2}$.

(3) To apply this or similar techniques to more general quadratics, we need to use a technique known as *completing the square*. Here, we rewrite:

$$Ax^2 + Bx + C = A(x + (B/2A))^2 + (C - B^2/4A)$$

The special case where $A = 1$ is given by:

$$x^2 + Bx + C = (x + (B/2))^2 + (C - (B/2)^2)$$

Note that the left-over constant term after completing the square is $-D/4A$ where D is the discriminant of the quadratic polynomial. In the case $A = 1$, when the polynomial has positive discriminant, this left-over term is negative, whereas when the polynomial has negative discriminant, this left-over term is positive. In the latter case, we can write it as the square of something. We would thus have written our original polynomial as $(x - \beta)^2 + \gamma^2$, whereupon we can make the substitution $\theta = \arctan((x - \beta)/\gamma)$ (or directly apply the integration formula).

1. The key idea of substitution

We are often faced with situations where the integrand is an algebraic expression that involves a squareroot sign. The key idea is to use a trigonometric substitution that converts the problem to a trigonometric integration. We then use the plethora of trigonometric identities to simplify this integral.

1.1. Substitutions involving $\sqrt{a^2 - x^2}$.

We first recall the following basic facts that will provide context for the trigonometric substitutions that follow:

(1) If $\theta = \arcsin(u)$, then $\sin\theta = u$ and $\cos\theta = \sqrt{1 - u^2}$. Note that it is the nonnegative squareroot because *cosine is nonnegative on the range of the arcsine function*.
(2) We have $\int dx/\sqrt{1 - x^2} = \arcsin(x)$. We obtained this result by noting that $\arcsin'(x) = 1/\sin'(\arcsin(x))$ and simplifying.
(3) In general, if we make the substitution $\theta = \arcsin(u)$ in an integration problem, then $du = \cos\theta\, d\theta$, $u = \sin\theta$, and $\sqrt{1 - u^2} = \cos\theta$.
(4) Even more generally, if we put $\theta = \arcsin(x/a)$ in an integration problem (with $a > 0$ a constant), then $dx = a\cos\theta\, d\theta$, $x = a\sin\theta$, and $\sqrt{a^2 - x^2} = a\cos\theta$. In reverse, $\sin\theta = x/a$, $\cos\theta = \sqrt{1 - (x/a)^2} = \sqrt{a^2 - x^2}/a$.

This brings us to the key idea of integration: if we see the expression $\sqrt{a^2 - x^2}$ in the integrand, we should consider the substituion $\theta = \arcsin(x/a)$, further getting $dx = a\cos\theta d\theta$, $x = a\sin\theta$, and $\sqrt{a^2 - x^2} = a\cos\theta$. For instance, consider:

$$\int \sqrt{a^2 - x^2}\, dx$$

Using the substitution $\theta = \arcsin(x/a)$, we get:

$$\int a\cos\theta(a\cos\theta d\theta) = \int a^2\cos^2\theta\, d\theta$$

We can simplify this further, using the well-memorized formula for the antiderivative of $\cos^2\theta$. We get:

$$a^2\left[\frac{\theta}{2} + \frac{\sin(2\theta)}{4}\right] = a^2\left[\frac{\theta + \sin\theta\cos\theta}{2}\right]$$

Putting $\theta = \arcsin(x/a)$, and using $\sin\theta = x/a$ and $\cos\theta = \sqrt{1 - (x/a)^2}$, we simplify and obtain:

$$\frac{a^2}{2}\arcsin(x/a) + \frac{x}{2}\sqrt{a^2 - x^2}$$

This should all be familiar to you, since it was a homework problem.

The idea works more generally for half-integer powers of $a^2 - x^2$. For instance, consider:

$$\int (a^2 - x^2)^{3/2}\, dx$$

After the trigonometric substitution, we obtain:

$$\int a^3 \cos^3\theta (a\cos\theta)\, d\theta$$

This reduces to integrating $\cos^4\theta$. We can now go a number of routes – we can use the reduction formula to reduce it to the integral of $\cos^2\theta$, or we can use a bunch of trigonometric identities using double angle formulas.

More generally:

$$\int (a^2 - x^2)^{n/2}\, dx$$

becomes, after the appropriate trigonometric substitution $\theta = \arcsin(x/a)$:

$$a^{n+1} \int \cos^{n+1}\theta\, d\theta$$

Note that this formula works for negative n as well. The particular case $n = -1$ is the familiar integration formula $\int dx/\sqrt{a^2 - x^2} = \arcsin(x/a)$.

1.2. **A slight complication.** Consider an integral of the form (as before, $a > 0$):

$$\int x^m (a^2 - x^2)^{n/2}\, dx$$

There are three cases of note:

(1) n is even and nonnegative: In this case, we can expand using the binomial theorem and integrate termwise.
(2) m is odd: In this case, we can make a u-substitution $u = a^2 - x^2$, and solve the integral in a purely algebraic fashion. We could also do the trigonometric substitution if we so desire, but to solve that problem we would end up doing an algebraic substitution back again..
(3) Other cases: We can use the trigonometric substitution $\theta = \arcsin(x/a)$ and simplify. We basically reduce to the case of integrating the product of a power of the sine function and a power of the cosine function.

1.3. **The two other key substitution ideas.** We will now state the two other key ideas for substitutions:

(1) An expression of the form $a^2 + x^2$ with a negative power or squareroot to it should suggest $\theta = \arctan(x/a)$, giving $dx = a\sec^2\theta\, d\theta$, $x = a\tan\theta$, and $a^2 + x^2 = a^2\sec^2\theta$. Also, $\sqrt{a^2 + x^2} = a\sec\theta$. Also, $\tan\theta = x/a$, $\sec\theta = \sqrt{1 + (x/a)^2} = \sqrt{a^2 + x^2}/a$. Here, we are using the fact that sec is positive on the range of the arc tangent function.
(2) An expression of the form $\sqrt{x^2 - a^2}$ should suggest $\theta = \operatorname{arcsec}(x/a) = \arccos(a/x)$, i.e., $x = a\sec\theta$, giving $dx = a\sec\theta\tan\theta\, d\theta$ and $\sqrt{x^2 - a^2} = a|\tan\theta|$. We cannot dispense with the absolute value sign because tangent is not positive throughout the range of the arc secant function (which is the same as the range of the arc cosine function).

We use this to calculate some important trigonometric integrals:

$$\int \frac{dx}{\sqrt{a^2 + x^2}}$$

We use the substitution $\theta = \arctan(x/a)$ and obtain:

$$\int \frac{a\sec^2\theta}{a\sec\theta}\, d\theta$$

After some cancellation, this reduces to:

$$\int \sec\theta\, d\theta$$

This gives:

$$\ln|\sec\theta + \tan\theta|$$

Note that $\tan\theta = x/a$, and $\sec\theta = \sqrt{1 + (x/a)^2}$, so we get:

$$\ln\left|\sqrt{1 + \frac{x^2}{a^2}} + \frac{x}{a}\right|$$

Note that, interestingly, the final answer can be written in a manner that is completely devoid of trigonometry. However, the trigonometric route was useful in obtain this answer.

Here's another example:

$$\int \sqrt{a^2 + x^2}\, dx$$

We use the substitution $\theta = \arctan(x/a)$ and obtain:

$$\int a\sec\theta a\sec^2\theta\, d\theta$$

This reduces to a^2 times the integral of $\sec^3\theta$, which we have seen using integration by parts. The answer is:

$$\frac{a^2}{2}\left[\sec\theta\tan\theta + \ln|sec\theta + \tan\theta|\right]$$

We can now substitute back and simplify, writing functions of θ in terms of x and a.

1.4. **Application to higher powers of $x^2 + a^2$.** Next, consider an integral of the form:

$$\int \frac{dx}{(x^2 + a^2)^{n/2}}$$

We use the substitution $\theta = \arctan(x/a)$, and simplify to obtain:

$$\frac{1}{a^{n-1}} \int \cos^{n-2}\theta\, d\theta$$

The special case $n = 2$ just gives $\theta/a = (1/a)\arctan(x/a)$. The case $n = 3$ gives:

$$\frac{1}{a^2} \int \cos\theta\, d\theta = (\sin\theta)/(a^2)$$

We can now rewrite $\sin\theta$ in terms of x and a.

The case $n = 4$ gives:

$$\frac{1}{a^3} \int \cos^2\theta\, d\theta$$

which we can solve using the well memorized integral of $\cos^2$, and then substitute back in terms of x and a.

Aside: quick as a fox. To get really good at these integration problems, it helps to memorize the way the substitutions typically work. But it also helps to memorize key integration results in a manner that they can be easily applied to problems directly. Thus, I recommend that, once you have a basic mastery of the methods, you memorize the integrals of various half-integer powers of $x^2 - a^2$, $x^2 + a^2$, and $a^2 - x^2$. This memorization should include remembering the final answer clearly, remembering the steps used to reach it, and being able to quickly apply the learned formula to specific numerical values.

Using triangles. If you find it hard to deal with ratios when doing trigonometric substitutions in forward and reverse, you may benefit from using right triangles. The book uses this approach. We'll briefly sketch it here, and you can see worked examples in the book.

For instance, when doing the u-substitution $\theta = \arcsin(x/a)$, consider a right triangle with hypotenuse a, base angle θ, and height x (so x is the side opposite θ). Now use the Pythagorean theorem to deduce that the other side is $\sqrt{a^2 - x^2}$. It is now possible to compute all the trigonometric functions for θ in terms of the sides of the triangle.

This approach is not really different from what we discussed, but some people find it more intuitive. Refer to examples in the book.

2. Interpretation of formulas in terms of homogeneous functions

This is optional material, in the sense that it will not be directly tested, but it may help you understand things.

Here, we briefly discuss the concept of homogeneous degree and how it can be used to obtain a qualitative understanding of some of the integration formulas.

Consider a function $F(x, a)$ of two variables. We say that F is homogeneous of degree d if, for any λ, we have:

$$F(\lambda x, \lambda a) = \lambda^d F(x, a)$$

A homogeneous function of degree zero is sometimes called *dimensionless*, and it depends only on the quotient x/a.

A *homogeneous polynomial* of degree d is a polynomial in which each monomial has total degree d in the two variables. A homogeneous polynomial of degree d is also a homogeneous function of degree d.

We also have the following:

- The zero function can be viewed as homogeneous of any positive degree, but more properly, it is just treated as an anomaly.
- If F_1 and F_2 are homogeneous of the same degree d, so is any *linear combination* $a_1F_1 + a_2F_2$ (unless that linear combination is identically the zero function), where a_1 and a_2 are real constants. In particular, $F_1 + F_2$ and $F_1 - F_2$ are homogeneous of degree d.
- If F_1 and F_2 are homogeneous of degrees d_1 and d_2 respectively, then $F_1 \cdot F_2$ is homogeneous of degree $d_1 + d_2$ and F_1/F_2 is homogeneous of degree $d_1 - d_2$.
- If F is homogeneous of degree d, then F^m (where the power denotes a pointwise power) is homogeneous of degree dm. Here m could be an integer or a rational number.
- Applying any function to something homogeneous of degree zero gives something homogeneous of degree zero.

Combining these two observations, we see that $1/\sqrt{a^2 - x^2}$ is homogeneous of degree -1, $1/(x^2 + a^2)$ is homogeneous of degree -2, and $(x^2 + a^2)^{3/2}$ is homogeneous of degree two. Now, we make the key observations relevant to the differentiation and integration formulas:

- Differentiation of a homogeneous function in x and a with respect to x gives a homogeneous function with degree one less.
- Conversely, integration of a homogeneous function in x and a with respect to x usually gives a homogeneous function with degree one more than the integrand, plus a constant.

  Note, please, that not every antiderivative expression is a homogeneous function. Rather, all we are saying is that *one* of the antiderivatives is homogeneous, so every antiderivative is a constant plus a homogeneous function. However, the *usual* methods we use to integrate will naturally yield the homogeneous antiderivative.
- The upshot is that when integrating some radically thing which is homogeneous in x and a of degree d, we will get something which is homogeneous in x and a of degree $d+1$. *Further*, all the parts that involve inverse trigonometric functions will be of the form a^{d+1} times some inverse trigonometric function (or variant) of x/a. Any expression involving logarithms should involve the logarithm of some function of x/a (i.e., should have degree zero).

Here are some examples (in all of which we assume $a > 0$):

- The integral of $1/\sqrt{a^2 - x^2}$ is $\arcsin(x/a)$. The integrand is homogeneous of degree -1, and the expression we get after integrating is homogeneous of degree 0.
- The integral of $1/(x^2 + a^2)$ is $(1/a)\arctan(x/a)$. The integrand is homogeneous of degree -2, and the expression we get after integrating is homogeneous of degree -1 – namely, it is the product of $1/a$ and the dimensionless quantity $\arctan(x/a)$.
- The integral of $1/\sqrt{x^2 + a^2}$ is $\ln[(x/a) + \sqrt{(x/a)^2 + 1}]$. The integrand is homogeneous of degree -1 and after integrating we get something that is homogeneous of degree 0.
- The integral of $\sqrt{a^2 - x^2}$ is $(a^2/2)\arcsin(x/a) + x\sqrt{a^2 - x^2}/2$. The integrand is homogeneous of degree one and after integrating we get something that is homogeneous of degree two – it is the sum of two terms each of which is homogeneous of degree two.

3. Alternative interpretation: inverse hyperbolic trigonometry

This section is optional, and is not officially part of the course, but is included to help offer another perspective to these integrations.

Recall that we did the integration:

$$\int \frac{dx}{\sqrt{x^2 + a^2}} = \ln\left[\frac{x}{a} + \sqrt{(x/a)^2 + 1}\right]$$

We did this integration using a trigonometric substitution and then using the formula for integrating the secant function. That, however, is *not* the *natural* approach to this problem. The natural approach is to consider the *arc hyperbolic sine* function, briefly discussed here.

Recall that sinh is a one-to-one function with domain and range $\mathbb{R}$. Thus, we can define an inverse function, which we denote $\sinh^{-1}$, on all of $\mathbb{R}$. If $\sinh x = y$, then we have:

$$\cosh^2 x = y^2 + 1$$

Since cosh is positive, we get $\cosh x = \sqrt{y^2 + 1}$, so $\exp(x)$ becomes $\sinh x + \cosh x$, which is $y + \sqrt{y^2 + 1}$. Thus, we get:

$$x = \ln[y + \sqrt{y^2 + 1}]$$

Interchanging the roles of x and y to get the explicit expression for $\sinh^{-1}$, we get $\sinh^{-1} x = x + \sqrt{x^2 + 1}$. (This was seen in one of the quiz problems).

Now, getting back to the integration problem (with $a > 0$):

$$\int \frac{dx}{\sqrt{x^2 + a^2}}$$

Put $t = \sinh^{-1}(x/a)$. Then $x = a\sinh t$, $dx = a\cosh t\,dt$, and we get:

$$\int \frac{a\cosh t\,dt}{\sqrt{a^2(\sinh^2 t + 1)}}$$

Using that $\sqrt{\sinh^2 t + 1} = \cosh t$, we get:

$$\int 1\,dt$$

which gives that the integral is $\sinh^{-1}(x/a)$. Now using the explicit expression for $\sinh^{-1}$ worked out above, we get the result indicated.

In general, any integration that involves a half-integer power of $a^2 + x^2$ is best done using inverse hyperbolic sine, and anything that involves an integer power (possibly negative) is best done using the arc tangent. However, as we have seen, it is possible (though messy) to do all these types of integrations using arc tangent alone, as long as we are prepared to integrate odd powers of secant using integration by parts. We'll stick to using only inverse circular trigonometry for this course.

If you want to learn more on hyperbolic trigonometry, go through Section 7.9 of the book, which we're not including in the syllabus. $\sinh^{-1}$ and other related functions are discussed on Pages 394 and 395 of the book.

4. Dealing with quadratics: square completion

4.1. The basics. We have looked at quadratics in the past, but we now need to consider them from a somewhat different perspective.

Given a quadratic function $f(x) := Ax^2 + Bx + C$ with $A \neq 0$, we can write:

$$f(x) = A\left(x + \frac{B}{2A}\right)^2 + \frac{4AC - B^2}{4A}$$

The expression $B^2 - 4AC$ is termed the *discriminant* of the quadratic, and we will, for convenience, denote it by the letter D. We thus have:

$$f(x) = A\left(x + \frac{B}{2A}\right)^2 - \frac{D}{4A}$$

The derivative is:

$$f'(x) = 2A\left(x + \frac{B}{2A}\right) = 2Ax + B$$

The only critical point is $x = -B/2A$. We now consider various cases:

(1) If $A > 0$, the quadratic goes to $+\infty$ as $x \to \pm\infty$, it is decreasing for x in $(-\infty, -B/2A)$ and it is increasing for x in $(-B/2A, \infty)$. The minimum is $-D/4A$, attained at $-B/2A$.
(2) If $A < 0$, the quadratic goes to $-\infty$ as $x \to \pm\infty$, it is increasing for x in $(-\infty, -B/2A)$, and it is decreasing for x in $(-B/2A, \infty)$. The maximum is $-D/4A$, attained at $-B/2A$.

We also see from the above that if $D < 0$, then the function has constant sign on all of $\mathbb{R}$, and never becomes zero. If $D = 0$, the function attains the value 0 at its vertex $-B/2A$ and has constant sign everywhere else. If $D > 0$, the function attains the value 0 at two distinct points. Note also that the symmetry of the graph about the line $x = -B/2A$ is clear from the context.

4.2. The upshot. The upshot of the above is that every quadratic function can be written as:

$$[\text{constant}] \times [\text{square of } (x - \text{ something})] + \text{constant}$$

For simplicity, we will assume, through a change of variable, that that $x-$ something is just x, i.e., that $B = 0$ in the original quadratic. We can do this change of variable. The upshot is that every quadratic can, after this transformation, be written as:

$$px^2 + q = p(x^2 + q/p)$$

There are now the following three possibilities relevant to integration situations:

(1) p and q are both positive or both negative: In this case, we can find a such that $a^2 = q/p$ and then use the substitution $\theta = \arctan(x/a)$.
(2) p and q have opposite signs: In this case, we can find a such that $a^2 = -q/p$. We can now put $\theta = \arcsin(x/a)$ (if that makes sense in the context) or put $\theta = \arccos(a/x)$ (if that makes sense in the context).
(3) $q = 0$: Here, integration poses no challenges.

4.3. **Completing the square: some examples.** For instance, consider:

$$\int \frac{dx}{x^2 + x + 1}$$

By the general discussion above, we can write this as:

$$\int \frac{dx}{(x + 1/2)^2 + 3/4}$$

We see that here, the second term is positive, and we obtain:

$$\int \frac{dx}{(x + 1/2)^2 + (\sqrt{3}/2)^2}$$

The trigonometric substitution is now clear: $\theta = \arctan{(x + 1/2)/(\sqrt{3}/2)}$. In this case, we can directly apply one of the antiderivative formulas (so we don't even need to formally do the substitution) and we get:

$$\frac{2}{\sqrt{3}} \arctan\left(\frac{x + (1/2)}{\sqrt{3}/2}\right) + C$$

Here is another example:

$$\int \frac{dx}{\sqrt{1 - 2x - x^2}}$$

We can use the square completion to obtain:

$$\int \frac{dx}{\sqrt{2 - (x + 1)^2}}$$

We can thus write it as:

$$\int \frac{dx}{\sqrt{(\sqrt{2})^2 - (x + 1)^2}}$$

We now put $\theta = \arcsin((x + 1)/\sqrt{2})$ and obtain, after simplification, that the integral is:

$$\arcsin\left(\frac{x + 1}{\sqrt{2}}\right) + C$$

Note that in both the above cases, we have been lucky in the sense that the square term had a coefficient of ± 1. Let's consider an example where it does not.

$$\int \frac{dx}{2x^2 + 3x + 4}$$

We can complete the square as:

$$\int \frac{dx}{2(x + (3/4))^2 + (4 - (9/8))}$$

We can now proceed, but it is usually easier if we take the coefficient of the square term outside the integral sign, obtaining, in this case:

$$\frac{1}{2} \int \frac{dx}{(x + (3/4))^2 + 23/16}$$

We now write $23/16 = (\sqrt{23}/4)^2$ and obtain the result in terms of the arc tangent function, namely:

$$\frac{2}{\sqrt{23}} \arctan\left(\frac{x + (3/4)}{\sqrt{23}/4}\right)$$

Mathematics is beautiful, but it is not always pretty.