HYPERBOLIC FUNCTIONS: AN INTRODUCTION

MATH 153, SECTION 55 (VIPUL NAIK)

Corresponding material in the book: Section 7.8.

What students should already know: The definition of the exponential function, the graphs of e^x and e^{-x}, the derivative and antiderivative of the exponential.

What students should definitely get: The definitions of hyperbolic sine and hyperbolic cosine, the fact that these are derivatives of each other, the graphs of these functions, the key identities involving hyperbolic sine and cosine and the general procedure of proving them.

What students should hopefully get: The analogy between sine/cosine and hyperbolic sine/cosine, a general comprehension of Osborne's rule.

Important note: We are skipping, for now, the material in the book titled "Catenary" and "Relation to the hyperbola $x^2 - y^2 = 1$" – though we might talk about it later. You are encouraged to read this material for your own curiosity.

Executive summary

(1) We define *hyperbolic cosine* $\cosh x := (e^x + e^{-x})/2$ and *hyperbolic sine* $\sinh x := (e^x - e^{-x})/2$. cosh is the *even part* of the exponentiation function (and in particular, is an even function) while sinh is the *odd part* of the exponentiation function (and in particular, is an odd function).
(2) cosh and sinh are derivatives of each other, and hence also antiderivatives of each other.
(3) cosh is even and positive, decreasing on $(-\infty, 0)$ and increasing on $(0, \infty)$, concave up throughout, goes to ∞ as $x \to \pm\infty$, and its local and absolute minimum value of 1 are attained at 0.
(4) sinh is odd, increasing on all of $\mathbb{R}$, negative and concave down on $(-\infty, 0)$, and positive and concave up on $(0, \infty)$. It passes through $(0, 0)$ where it has its unique point of inflection. Note that at $(0, 0)$, the derivative takes its minimum value, which is 1. In this important respect, the graph does *not* look like x^3, where we have a horizontal tangent at $x = 0$.
(5) $\cosh^2 x - \sinh^2 x = 1$. A lot of the identities involving hyperbolic sine and hyperbolic cosine look very similar to the corresponding identities involving the trigonometric (circular) sine and cosine. In fact, we can move back and forth between the circular and the hyperbolic using the following rule: change the sign in front of any term that involves a product of two sine terms. This rule is termed *Osborne's rule*.

1. The hyperbolic sine and cosine

1.1. Definition and the key derivative property.

We define the following two functions from $\mathbb{R}$ to $\mathbb{R}$.

The *hyperbolic cosine*, denoted cosh, is defined as:

$$\cosh(x) := \frac{e^x + e^{-x}}{2}$$

The *hyperbolic sine*, denoted sinh, is defined as:

$$\sinh(x) := \frac{e^x - e^{-x}}{2}$$

The key facts about these two functions are:

(1) $\cosh' = \sinh$. In other words, $\int \sinh(x)\,dx = \cosh x + C$.
(2) $\sinh' = \cosh$. In other words, $\int \cosh(x)\,dx = \sinh x + C$.
(3) $\cosh'' = \cosh$ and $\sinh'' = \sinh$: These follow from the above two.

1.2. **Even and odd.** Hyperbolic cosine is an *even* function, i.e., $\cosh(-x) = \cosh(x)$. This is because it is obtained by *averaging* the value of e^x and e^{-x}. Similarly, hyperbolic sine is an *odd* function, i.e., $\sinh(-x) = -\sinh x$.

In this respect, hyperbolic cosine behaves like cosine, and hyperbolic sine behaves like sine.

Recall, from a long time ago, that for any function f defined on $\mathbb{R}$, we can write f as the sum of an even function f_e and an odd function f_o in a *unique* way. Here, we define:

$$f_e(x) := \frac{f(x) + f(-x)}{2}$$

and

$$f_o(x) := \frac{f(x) - f(-x)}{2}$$

The even function f_e is termed the *even part* of f and the function f_o is termed the *odd part* of f. Further:

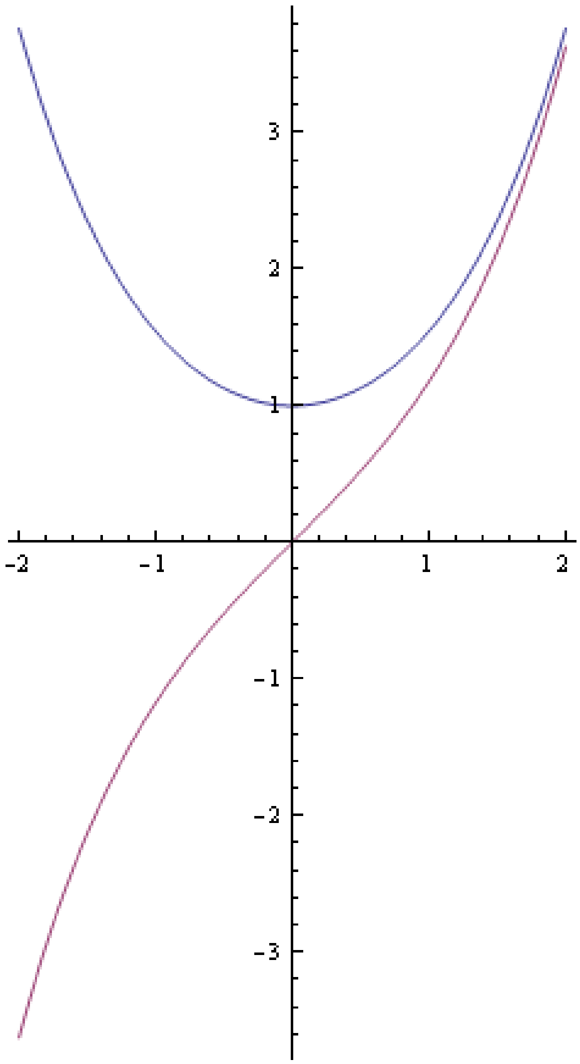
$$f(x) = f_e(x) + f_o(x), \qquad f(-x) = f_e(x) - f_o(x)$$

How does this relate to cosh and sinh? If we take $f = \exp$, i.e., $f(x) := e^x$, then the even part of f is cosh and the odd part of f is sinh. What we have done essentially is to split the exponentiation function additively into its even and odd parts. In particular, $\exp(x) = \cosh(x) + \sinh(x)$ and $\exp(-x) = \cosh(x) - \sinh(x)$.

1.3. **Graphing the functions.** We make the following observations in sequence:

(1) cosh is always positive, while sinh is positive for $x > 0$, negative for $x < 0$, and zero at $x = 0$. Thus, while the graph of cosh is completely above the x-axis, the graph of sinh is above the x-axis for $x > 0$ and below the x-axis for $x < 0$. Also, it passes through the origin.
(2) The derivative of cosh is sinh, which is positive for $x > 0$ and negative for $x < 0$. Thus, the graph of cosh is decreasing on $(-\infty, 0)$ and increasing on $(0, \infty)$. The derivative of sinh is cosh, which is always positive. Thus the graph of sinh is increasing on all of $\mathbb{R}$.
(3) The second derivative of cosh is cosh, which is always positive. Thus, the graph of cosh is concave up on all of $\mathbb{R}$. The second derivative of sinh is sinh, which is positive on $x > 0$ and negative on $x < 0$. Thus, the graph of sinh is concave down on $x < 0$ and concave up on $x > 0$.
(4) The summary: cosh is an even function that approaches $+\infty$ as $x \to \pm\infty$, is concave up throughout, decreases on $(-\infty, 0)$, increases on $(0, \infty)$, and has a unique local and absolute minimum at 0 with value 1. sinh is an odd function that approaches $-\infty$ at $-\infty$ and $+\infty$ at $+\infty$, increases throughout, is concave down on $(-\infty, 0)$, is concave up on $(0, \infty)$, and has a unique point of inflection at 0 with value 0. Note that this is *not* a horizontal tangent-type point of inflection of the x^3 creed. Rather, the tangent line to this is $y = x$. The *minimum value* of the derivative of sinh is 1, attained at 0.

Here are the graphs of the two functions:



In addition to everything we've noted so far, it's also true that the graphs of cosh and sinh are asymptotic to each other as $x \to +\infty$ but *not* as $x \to -\infty$. To see this, note that $\cosh x - \sin h = \exp(-x)$, which tends to 0 as $x \to +\infty$ and tends to $+\infty$ as $x \to -\infty$.

1.4. **Basic identities.**

$$\begin{aligned}
\cosh^2 x - \sinh^2 x &= 1 \\
\cosh(x+y) &= \cosh x \cosh y + \sinh x \sinh y \\
\sinh(x+y) &= \sinh x \cosh y + \cosh x \sinh y \\
\cosh(x-y) &= \cosh x \cosh y - \sinh x \sinh y \\
\sinh(x-y) &= \sinh x \cosh y - \cosh x \sinh y \\
\sinh(2x) &= 2 \sinh x \cosh x \\
\cosh(2x) &= 2\cosh^2 x - 1 = 2\sinh^2 x + 1 = \cosh^2 x + \sinh^2 x
\end{aligned}$$

All of these identities can be proved by plugging in the definitions of cosh and sinh in terms of exp and then using the properties of exp to simplify the expression on both sides. For instance:

$$\cosh^2 x - \sinh^2 x = \left[\frac{e^x + e^{-x}}{2}\right]^2 - \left[\frac{e^x - e^{-x}}{2}\right]^2 = \frac{e^{2x} + 2 + e^{-2x}}{4} - \frac{e^{2x} - 2 + e^{-2x}}{4} = \frac{2+2}{4} = 1$$

2. Relating the hyperbolic sine and cosine to the circular sine and cosine

In this section, we use the term *circular* sine and cosine to refer to the usual sine and cosine.

The hyperbolic sine and cosine have very similar behavior to the circular sine and cosine in the following important respects:

(1) *The mutual derivative relationship, along with the even-odd nexus*: The hyperbolic sine and cosine are derivatives of each other, with the former being even and the latter being odd. The circular cosine is the derivative of the circular sine and the circular sine is the *negative* derivative of the circular cosine.

(2) *Very similar identities* such as $\cos^2 + \sin^2 = 1$ being replaced by $\cosh^2 - \sinh^2 = 1$, as well as similar identities for sums and doubles.

On the other hand, the graphs of the functions look very different. sin and cos are periodic functions with a period of 2π, while sinh and cosh barely repeat (sinh is one-to-one and cosh restricts to a one-to-one function on the nonnegative reals). So what is really going on?

The answer to this is beyond our current scope, but it in fact has something to do with complex numbers. It turns out that we can extend all these definitions to complex inputs, and with that extended definition, we have $\cos(x) = \cosh(ix)$ and $\sin(x) = \sinh(ix)/i$, where i is a non-real squareroot of -1. The fact that $i^2 = -1$ is the key reason for both the similarities and differences.

This gives rise to a rule called *Osborne's rule*: To convert an identity involving the circular sine and cosine to a corresponding identity involving hyperbolic sine and cosine, change the sign on all terms that involve the product of two sines. For instance, in the identity $\cos^2 x + \sin^2 x = 1$, the $\sin^2$ term is the product of two sin terms, so the sign on this is changed. (The precise rule is more sophisticated, but this is good enough). It is a good exercise to use this rule to relate each of the hyperbolic sine/cosine identities and the corresponding circular sine/cosine identity.