TRIGONOMETRIC LIMITS AND DERIVATIVES

MATH 152, SECTION 55 (VIPUL NAIK)

Corresponding material in the book: Section 3.6.

Difficulty level: Easy to moderate, particularly if you remember corresponding stuff from AP level calculus.

Covered in class?: Yes, but not necessarily all examples.

What students should definitely get: The key trigonometric limits. The key differentiation formulas for trigonometric functions.

What students should eventually get: Techniques for computing limits and derivatives involving composites of trigonometric functions with each other and with polynomial and rational functions.

Executive summary

Words ...

(1) The following three important limits form the foundation of trigonometric limits: $\lim_{x \to 0} (\sin x)/x = 1$, $\lim_{x \to 0} (\tan x)/x = 1$, and $\lim_{x \to 0} (1 - \cos x)/x^2 = 1/2$.

(2) The derivative of sin is cos, the derivative of cos is $-\sin$. The derivative of tan is $\sec^2$, the derivative of cot is $-\csc^2$, the derivative of sec is $\sec \cdot \tan$, and the derivative of csc is $-\csc \cdot \cot$.

(3) The second derivative of any function of the form $x \mapsto a \sin x + b \cos x$ is the negative of that function, and the fourth derivative is the original function.

Actions ...

(1) Substitution is one trick that we use for trigonometric limits: we translate $\lim_{x \to c}$ to $\lim_{h \to 0}$ where $x = c + h$.

(2) Multiplicative splitting, chaining, and stripping are some further tricks that we often use.

(3) For derivatives of functions that involve composites of trigonometric and polynomial functions, we *have* to use the chain rule as well as rules for sums, differences, products, and quotients when simplifying expressions.

1. Some critical trigonometric limits

Sinning by degrees is costly. For all applications of trigonometry to limits and calculus, *all angles are expressed in radians*. The radian measurement is the natural measurement for an angle.

1.1. The sinc function.
There are many other minor matters related to trigonometric functions that we need to address, but for now, let's get back and focus on one very important function – the so-called sinc function. This function is defined as $(\sin x)/x$ for $x \neq 0$.

Now, the function isn't defined at $x = 0$, and it isn't immediately clear what the limit is. Because when we just try to substitute the value at 0, we get a 0/0 form. So it seems we need some angel to come and help us out.

Anyway, here's what the angel tells us:

$$\lim_{x \to 0} \frac{\sin x}{x} = 1$$

1.2. Demystifying our angel.
For an acute angle θ, $\sin \theta$ is the vertical height (the y-coordinate of the point we get after rotating counter-clockwise by an angle of θ from $(1, 0)$ along the unit circle, while θ is the arc length. You can see, from this picture, that the arc length θ is greater than $\sin \theta$. That's because $\sin \theta$ falls straight down while the arc moves both horizontally and vertically. However, and this is the crucial

point – as θ gets smaller and smaller, you see that the arc length and the vertical line seem to get closer and closer. And this suggests that, perhaps, as θ tends to zero, $\sin\theta/\theta$ tends to 1.

Now, this is hardly a proof, because two numbers getting really close does not necessarily mean that their ratio tends to 1. But it is suggestive. So, with this suggestivity, let's believe that the limit, as x tends to 0, of the fraction $\sin x/x$ is 1.

1.3. **The substitution idea.** We know that $\lim_{x\to0}(\sin x)/x = 1$. More generally, it is true that if f is continuous at c and $f(c) = 0$, then as $x \to c$, we have:

$$\lim_{x\to c} \frac{\sin(f(x))}{f(x)} = 1$$

For instance:

$$\lim_{x\to0} \frac{\sin(x^2)}{x^2} = 1$$

Similarly:

$$\lim_{x\to0} \frac{\sin(2x)}{2x} = 1$$

and:

$$\lim_{x\to0} \frac{\sin(x/2)}{x/2} = 1$$

On the other hand, if we consider the limit:

$$\lim_{x\to0} \frac{\sin(x + (\pi/3))}{x + (\pi/3)}$$

This limit is not 1 – the inner expression does *not* go to 0 as x goes to 0.

1.4. **Chaining limit computations.** Let's consider the computation:

$$\lim_{x\to0} \frac{\sin(\sin x)}{x}$$

This is a special case of a more general limit computation that you have seen in Question 2 of the October 4 quiz. Let's first do this specific example, and then return to how it relates to that question.

For the specific example, we note that the limit in question involves a composite function. For such problems, we typically *chain* the limit by multiplying and dividing by the inner function. We get:

$$\lim_{x\to0} \frac{\sin(\sin x)}{\sin x} \frac{\sin x}{x}$$

We now use that the limit of products is the product of limits, and obtain:

$$\lim_{x\to0} \frac{\sin(\sin x)}{\sin x} \lim_{x\to0} \frac{\sin x}{x}$$

The second limit is clearly 1. The first limit is also 1, because it is of the form $[\sin(f(x))]/f(x)$ where $f(x) \to 0$.

Now, the quiz question was that if $\lim_{x\to0} g(x)/x = A \neq 0$ with g continuous, then what is $\lim_{x\to0} g(g(x))/x$? The same chaining idea applies:

$$\lim_{x\to0} \frac{g(g(x))}{x} = \lim_{x\to0} \frac{g(g(x))}{g(x)} \frac{g(x)}{x}$$

We again split the limit multiplicatively, and argue that both component limits are A. For one of them, we have to argue that as $x \to 0$, $g(x) \to 0$ – an argument that we make in a somewhat indirect fashion. Go back to the quiz solution for more.

1.5. **Easier chainings.** Here are some easier examples:

$$\lim_{x \to 0} \frac{\sin(mx)}{x} = m$$

Here, we chain via mx.

$$\lim_{x \to 0} \frac{\sin(mx^n)}{x^n} = m$$

where n is positive.

1.6. **The $(1 - \cos x)/x^2$ limit.** We now show another fundamentally important trigonometric limit:

$$\lim_{x \to 0} \frac{1 - \cos x}{x^2} = \frac{1}{2}$$

We first show how to obtain this limit by multiplying both numerator and denominator by $1 + \cos x$. We get:

$$\lim_{x \to 0} \frac{1 - \cos^2 x}{x^2(1 + \cos x)}$$

The $1 + \cos x$ in the denominator pulls out by evaluation, and we get:

$$\frac{1}{2} \lim_{x \to 0} \frac{1 - \cos^2 x}{x^2}$$

We now use $1 - \cos^2 x = \sin^2 x$ and get:

$$\frac{1}{2} \lim_{x \to 0} \frac{\sin^2 x}{x^2}$$

This becomes:

$$\frac{1}{2} \lim_{x \to 0} \left(\frac{\sin x}{x} \right)^2$$

The limit of the square is the square of the limit (basically, a special case of the fact that the limit of the product is the product of the limit, so the inner limit is 1, and we get a $1/2$.

We can also obtain this limit using a double angle formula as described below:

The idea here is to use the identity we saw last time, which was that $\cos 2A = 1 - 2 \sin^2 A$. So $1 - \cos 2A = 2 \sin^2 A$. Here $A = x/2$, so we get:

$$\lim_{x \to 0} \frac{2 \sin^2(x/2)}{x^2}$$

We can pull the 2 out, and we get $\sin^2(x/2)/x^2$ inside. Now, the thing with calculating these limits is that we only know how to calculate the limit of the form $\sin \theta/\theta$, and here, $\theta = x/2$. So, we rewrite the denominator as $4(x/2)^2$, and we pull out the 4, so we get:

$$\frac{1}{2} \lim_{x \to 0} \left(\frac{\sin(x/2)}{(x/2)} \right)^2$$

Now, using the *limit of product equals product of limits* meme, we get:

$$\frac{1}{2} \lim_{x \to 0} \frac{\sin(x/2)}{(x/2)} \lim_{x \to 0} \frac{\sin(x/2)}{(x/2)}$$

Now, in both cases, we have $x \to 0$, so $x/2 \to 0$, so both limits are 1, and hence, our final answer is $1/2$.

The $\tan x/x$ **limit.** Let's now calculate the limit:

$$\lim_{x\to 0} \frac{\tan x}{x}$$

What we do is to write $\tan x = \sin x/\cos x$:

$$\lim_{x\to 0} \frac{\sin x}{x \cos x}$$

We split this as a product:

$$\lim_{x\to 0} \frac{1}{\cos x} \lim_{x\to 0} \frac{\sin x}{x}$$

Both limits are 1, so the overall limit is 1.

1.7. **Corollaries.** We can now state some easy corollaries of the above results:

$$\lim_{x\to 0} \frac{1 - \cos(mx)}{x^2} = \frac{m^2}{2}$$

We obtain by chaining via $(mx)^2$.
Similarly:

$$\lim_{x\to 0} \frac{\tan(mx)}{x} = m$$

1.8. **Substitution that involves translation.** There is a *substitution of variables* trick that we can use for computing limits. When we were computing limits for rational functions, we never really needed that trick, primarily because we knew how to handle rational functions anyway. But this trick comes in useful for trigonometric functions.

The trick is:

$$\lim_{x\to c} f(x) = \lim_{h\to 0} f(c + h)$$

Similarly:

$$\lim_{x\to c^+} f(x) = \lim_{h\to 0^+} f(c + h)$$

and:

$$\lim_{x\to c^-} f(x) = \lim_{h\to 0^-} f(c + h) = \lim_{h\to 0^+} f(c - h)$$

Why is this useful for trigonometric functions? Because for trigonometric functions, the *only* nontrivial limit that we know is the one I just told you: $\lim_{x\to 0} \frac{\sin x}{x} = 1$. So, we need to basically use this for any nontrivial limit that we need to compute. For instance, consider the limit:

$$\lim_{x\to \pi/4} \frac{\sin x - (1/\sqrt{2})}{x - \pi/4}$$

Now, we want to change the thing that's limiting to h, so we rewrite this as:

$$\lim_{h\to 0} \frac{\sin(\pi/4 + h) - (1/\sqrt{2})}{h}$$

We simplify the numerator using the $\sin(A + B)$ formula, which we know is $\sin A \cos B + \cos A \sin B$, and we get:

$$\lim_{h\to 0} \frac{(1/\sqrt{2})(\sin(h) + \cos(h) - 1)}{h}$$

We take out the $1/\sqrt{2}$ factor and now try to split the inner limit additively:

4

$$\frac{1}{\sqrt{2}} \left(\lim_{h \to 0} \frac{\sin h}{h} - \lim_{h \to 0} \frac{1 - \cos h}{h} \right)$$

The first limit is 1. For the second limit, it can be written as $\lim_{h \to 0} h \cdot \lim_{h \to 0} (1 - \cos h)/h^2 = 0(1/2) = 0$. So the second limit is 0. Overall, the limit is $1/\sqrt{2}$.

We'll see this same calculation a little later when we try to calculate the derivative of the sin function.

2. STRIPPING: A SNEAK PEEK

We will cover this somewhat delicate operation a little later in 153, when students are more mature (in terms of having seen more kinds of functions) and can handle the issue with the requisite care. But some of you may be equipped to use this approach to compute trigonometric limits more intuitively and rapidly now. [NOTE: To my knowledge, this particular intuitive approach to stripping is not found in any high school or college calculus text and can be considered my own invention, though most people who compute limits on a regular basis do this all the time.]

To motivate stripping, let us look at a fancy example:

$$\lim_{x \to 0} \frac{\sin(\tan(\sin x))}{x}$$

This is a composite of three functions, so if we want to chain it, we will chain it as follows:

$$\lim_{x \to 0} \frac{\sin(\tan(\sin x))}{\tan(\sin x)} \frac{\tan(\sin x)}{\sin x} \frac{\sin x}{x}$$

We now split the limit as a product, and we get:

$$\lim_{x \to 0} \frac{\sin(\tan(\sin x))}{\tan(\sin x)} \lim_{x \to 0} \frac{\tan(\sin x)}{\sin x} \lim_{x \to 0} \frac{\sin x}{x}$$

Now, we argue that each of the inner limits is 1. The final limit is clearly 1. The middle limit is 1 because the inner function $\sin x$ goes to 0. The left most limit is 1 because the inner function $\tan(\sin x)$ goes to 0. Thus, the product is $1 \times 1 \times 1$ which is 1.

If you are convinced, you can further convince yourself that the same principle applies to a much more convoluted composite:

$$\lim_{x \to 0} \frac{\sin(\sin(\tan(\sin(\tan(\tan x)))))}{x}$$

However, *writing that thing out takes loads of time.* Wouldn't it be nice if we could just strip off those sins and tans? In fact, we can do that.

The key stripping rule is this: *in a multiplicative situation* (i.e. there is no addition or subtraction happening), if we see something like $\sin(f(x))$ or $\tan(f(x))$, and $f(x) \to 0$ in the relevant limit, then we can strip off the sin or tan. In this sense, both sin and tan are *strippable* functions. A function g is strippable if $\lim_{x \to 0} g(x)/x = 1$.

The reason we can strip off the sin from $\sin(f(x))$ is that we can multiply and divide by $f(x)$, just as we did in the above examples.

Stripping can be viewed as a special case of the l'Hopital rule as well, but it's a much quicker shortcut in the cases where it works.

Thus, in the above examples, we could just have stripped off the sins and tans all the way through.

Here's another example:

$$\lim_{x \to 0} \frac{\sin(2 \tan(3x))}{x}$$

As $x \to 0$, $3x \to 0$, so $2 \tan 3x \to 0$. Thus, we can strip off the outer sin. We can then strip off the inner tan as well, since its input $3x$ goes to 0. We are thus left with:

$$\lim_{x \to 0} \frac{2(3x)}{x}$$

Cancel the x and get a 6. We could also do this problem by chaining or the l'Hopital rule, but stripping is quicker and perhaps more intuitive.

Here's yet another example:

$$\lim_{x \to 0} \frac{\sin(x \sin(\sin x))}{x^2}$$

As $x \to 0$, $x \sin(\sin x) \to 0$, so we can strip off the outermost sin and get:

$$\lim_{x \to 0} \frac{x \sin(\sin x)}{x^2}$$

We cancel a factor of x and get:

$$\lim_{x \to 0} \frac{\sin(\sin x)}{x}$$

Two quick sin strips and we get x/x, which becomes 1.

Yet another example:

$$\lim_{x \to 0} \frac{\sin(ax) \tan(bx)}{x}$$

where a and b are constants. Since this is a multiplicative situation, and $ax \to 0$ and $bx \to 0$, we can strip the sin and tan, and get:

$$\lim_{x \to 0} \frac{(ax)(bx)}{x}$$

This limit becomes 0, because there is a x^2 in the numerator and a x in the denominator, and cancellation of one factor still leaves a x in the numerator.

Here is yet another example:

$$\lim_{x \to 0} \frac{\sin^2(ax)}{\sin^2(bx)}$$

where a, b are nonzero constants. We can pull the square out of the whole expression, strip the sins in both numerator and denominator, and end up with a^2/b^2.

When you can't strip. The kind of situations where we are not allowed to strip are where the expression $\sin(f(x))$ is not just multiplied but is being added to or subtracted from something else. For instance, in order to calculate the limit:

$$\lim_{x \to 0} \frac{x - \sin x}{x^3}$$

stripping off the sin would be a sin. The intuition behind what is wrong with this will have to wait till next quarter.

What about $1 - \cos(f(x))$? If we see something like $1 - \cos(f(x))$ with $f(x) \to 0$ in the limit, then we know that it "looks like" $f(x)^2/2$ and we can replace it by $f(x)^2/2$. This can be thought of as a sophisticated version of stripping. For instance:

$$\lim_{x \to 0} \frac{(1 - \cos(5x^2))}{x^4}$$

Here $f(x) = 5x^2$, so the numerator is like $(5x^2)^2/2$, and the limit just becomes 25/2.

3.1. Differentiation formulas. The differentiation formulas are as follows:

$$
\begin{aligned}
\sin' &= \cos \\
\cos' &= -\sin \\
\tan' &= \sec^2 \\
\sec' &= \sec \cdot \tan \\
\cot' &= -\csc^2 \\
\csc' &= -\csc \cdot \cot
\end{aligned}
$$

We can obtain all these just from the differentiation formulas for sin and cos, using the quotient rule:

$$\frac{d}{dx}[\sin x] = \cos x$$

and:

$$\frac{d}{dx}[\cos x] = \sin x$$

In fact, the differentiation formula for cos can be obtained from that of sin using the chain rule as follows:

$$\frac{d}{dx}[\cos x] = \frac{d}{dx}[\sin((\pi/2) - x)] = -\cos((\pi/2) - x) = -\sin x$$

So, in order to obtain all these formulas, all we need to do is obtain the differentiation formula for sin. The derivation of this formula is given below. Essentially, it uses the fact that $\lim_{h \to 0}(\sin h)/h = 1$:

$$
\begin{aligned}
&\lim_{h \to 0} \frac{\sin(x+h) - \sin x}{h} \\
&= \lim_{h \to 0} \frac{\sin x \cos h + \cos x \sin h - \sin x}{h} \\
&= \sin x \lim_{h \to 0} \frac{\cos h - 1}{h} + \cos x \lim_{h \to 0} \frac{\sin h}{h} \\
&= \sin x \cdot 0 + \cos x \cdot 1 \\
&= \cos x
\end{aligned}
$$

The fact that $(\cos h - 1)/h \to 0$ can be deduced by multipled by $1 + \cos h$ and simplifying. Basically, $\cos h - 1$ goes to 0 at the rate of h^2, and the denominator is only h.
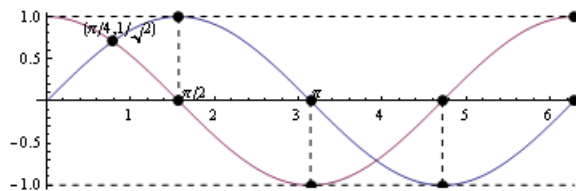
3.2. Graphical interpretation of these derivatives.

The fascinating thing about the sin function is that its derivative looks just like the function – except it's shifted over by $\pi/2$. Let's see what this means graphically.

At the point $x = 0$, the derivative is $\cos 0 = 1$. This means that the tangent is the line $y = x$. And indeed, close to the point 0, the sine graph looks a lot like the line $y = x$. However, there's an added subtlety that you may not be able to appreciate right now but will be able to later. The graph of the sine function on the right of the point 0 falls slightly on the lower side of the $y = x$ line, and the graph to the left of 0 falls slightly on the upper side of the line. In other words, the curve actually passes from one side of the tangent line at 0 to the other side.

This is a very unusual situation, because most of the time that you think of tangent lines, the graph of the function *close to the point* lies entirely on one side of the tangent line. But here, the graph is *crossing*

the tangent line. These kinds of points are called *points of inflection*, and believe it or not, for a twice differentiable function, a point of inflection is a point where the *second* derivative is zero (though there could be points where the second derivative is zero that are not points of inflection – funny things could happen with the *third* derivative). We'll talk more about inflections later.

Anyway, so at zero, the derivative is 1. What happens as x increases from 0 to $\pi/2$? The cosine function keeps decreasing form 1 to 0. But since the cosine function is positive, the tangent to the sine function is still upward-sloping, so the sine function is increasing, but the slope of the tangent line is falling. Finally at $x = \pi/2$, the sine function reaches its peak of 1, and its derivative, the cosine function, becomes 0.

So here's the beautiful things you see, and for which you now have a powerful explanation based on the derivative:

(1) The points where the sine function attains its maxima and minima are precisely the points where the cosine function is 0. Namely, these points are odd multiples of $\pi/2$. The points where the cosine function attains its maxima and minima are precisely the points where the sine function is 0. Namely, these points are the multiples of π.

(2) The regions where the sine function is increasing (respectively, decreasing) are the same as the regions where the cosine function is positive (respectively, negative). The regions where the cosine function is increasing (respectively, decreasing) are the same as the regions where the sine function is negative (respectively, positive).

3.3. **We've done it!** What I've given you is the complete toolkit with which you can calcuate any derivative that involves a mix of trigonometric, polynomial, and rational functions. Let's consider an example: $\sin(x^2)$.

How do you think of this function? To evaluate this function at a particular value of x, what you first do is calculate x^2, and then apply the sin function to that. So, you're doing two operations in sequence, whereby, you're feeding the output of one operation as the input of the other. So what you're doing is essentially *function composition*.

Which brings us back to the chain rule. Remember the way the chain rule works? We set $v = x^2$, first differentiate the function with respect to v, and then multiply by dv/dx. So, we get that the derivative is $(\cos(x^2)) \cdot 2x$.

For composites of trigonometric functions and polynomial, the chain rule is *indispensable* – if you don't want to use the chain rule, there's no way of calculating the derivative without *actually calculating it using the limit definition*. In the case of polynomials and rational functions, the chain rule was a convenience. Now it's a necessity.

3.4. **Periodicity of the sequence of derivative functions.** The derivative of sin is cos. The derivative of cos is $-\sin$. The derivative of $-\sin$ is $-\cos$. The derivative of $-\cos$ is sin.

In other words, the sin function equals its *fourth* derivative. If we consider the sequence of derivative functions of sin, this *sequence* has a *period* of four. (We'll formally define sequence and period of a sequence in 153, but you know what this means). Thus, for instance, the 101th derivative of sin is the same as the first derivative (because the remainder on dividing 101 by 4 is 1) and is thus cos.

[Aside: This is significant in many ways. For instance, when we study antidifferentiation (indefinite integration) we'll notice that the above basically tells us that we can keep taking antiderivatives of sin or cos and still remain sin or cos (up to a plus or minus). This is significant when, for instance, we study integration by parts. There, we will choose sin or cos as the *part to repeatedly integrate* precisely because repeated integration does not increase the complexity of the function.]

3.5. **Second derivative same with a minus sign.** Also note that for the sin function, the second derivative is the *negative* of the function. This is also true for the cos function. It is also true of all *linear combinations* of sin and cos, i.e., all functions of the form $x \mapsto a \sin x + b \cos x$ where $a, b \in \mathbb{R}$.[1] For any function f of this form, $f''(x) = -f(x)$.

What's so special about this fact? *This is the reason why trigonometric functions, specifically linear combinations of* sin *and* cos*, arise in nature.* What happens is that some basic physical/biological/chemical/ecological law or constraint forces the solution function we have to satisfy the equation $f''(x) = -\omega^2 f(x)$ or something

---

[1]Formally, this is the vector space of functions generated by sin and cos.

like that. Now, functions of the above form (with appropriate scaling) pop up naturally. More on this when we study differential equations.