REVIEW SHEET FOR MIDTERM 2: ADVANCED

MATH 152, SECTION 55 (VIPUL NAIK)

This is the part of the review sheet that we will concentrate on during the review session.

1. Left-overs from differentiation basics

1.1. **Derivative as rate of change.** No error-spotting exercises

1.2. **Implicit differentiation.** No error-spotting exercises

2. Increase/decrease, maxima/minima, concavity, inflection, tangents, cusps, asymptotes

2.1. **Rolle's, mean value, increase/decrease, maxima/minima.** Error-spotting exercises

(1) If a function f has a local maximum at a point c in its domain, then f is increasing on the immediate left of c and decreasing on the immediate right of c.

(2) Consider the function:

$$f(x) := \{ \begin{array}{rl} x^3 - 12x + 14, & x \le 1 \\ x^2 - 6x + 8, & x > 1 \end{array}$$

The derivative is:

$$f'(x) = \{ \begin{array}{rl} 3x^2 - 12, & x \le 1 \\ 2x - 6, & x > 1 \end{array}$$

The solutions for $f'(x) = 0$ are $x = -2$ and $x = 2$ (for the $x \le 1$ case) and $x = 3$ (the $x > 1$ case). Thus, the critical points are at $x = -2$, $x = 2$, and $x = 3$.

(3) Consider the function:

$$f(x) := x^4 - x + 1$$

The derivative is:

$$f'(x) = 4x^3 - 1$$

Solve $f'(x) = 0$ and we get $x = (1/4)^{1/3}$. Thus, f has a local maximum at $x = (1/4)^{1/3}$. The local maximum value is:

$$4((1/4)^{1/3})^3 - 1$$

which is 0.

(4) Consider the function

$$f(x) := \frac{1}{x^3 - 1}$$

The derivative is:

$$f'(x) = \frac{3x^2}{(x^3 - 1)^2}$$

The derivative is zero at $x = 0$, so that gives a critical point. Also, the derivative is undefined at $x = 1$, so that gives another critical point for f.

(5) An everywhere differentiable function f on $\mathbb{R}$ has critical points at 2, 5, and 9 with corresponding function values 11, 16, and 3 respectively. Thus, the absolute maximum value of f is 16 and the absolute minimum value is 3.

2.2. **Concave up/down and points of inflection.** Error-spotting exercises ...

(1) Consider the function

$$f(x) := 3x^5 - 5x^4 + 12x + 17$$

The derivative is:

$$f'(x) = 15x^4 - 20x^3 + 12$$

The second derivative is:

$$f''(x) = 60x^3 - 60x^2$$

The zeros of this are $x = 0$ and $x = 1$. The function thus has points of inflection at the points on the graph corresponding to $x = 0$ and $x = 1$.

(2) To check whether a critical point for the first derivative gives a point of inflection for the graph of the function, we need to check the sign of the third derivative. If the third derivative is nonzero, we get a point of inflection. If the third derivative is zero, then we *do not* get a point of inflection.

2.3. **Tangents, cusps, and asymptotes.** Cute fact: Rational functions are asymptotically polynomial, and the polynomial to which a given rational function is asymptotic (both directions) is obtained by doing long division and looking at the quotient. If the degree of the numerator is one more than that of the denominator, we get an oblique (linear) asymptote. If the numerator and denominator have equal degree, we get a horizontal asymptote (both directions) with nonzero value. If the numerator has smaller degree, the x-axis is the horizontal asymptote (both directions).

Error-spotting exercises

(1) If $\lim_{x\to\infty} f(x) = L$ with L a finite number, then $\lim_{x\to\infty} f'(x) = 0$.
(2) If $\lim_{x\to\infty} f'(x) = 0$, then $\lim_{x\to\infty} f(x) = L$, with L a finite number.
(3) If f' has a vertical tangent at a point a in its domain, then f has a point of inflection at $(a, f(a))$.
(4) If f' has a vertical cusp at a point a in its domain, then f has a local extreme value at a.
(5) Suppose f and g are functions defined on all of $\mathbb{R}$. Suppose f has a vertical tangent at a point a in its domain and g has a vertical tangent at a point b in its domain. Then $f + g$ has a vertical tangent at $a + b$ and $f - g$ has a vertical tangent at $a - b$.
(6) Suppose f and g are functions, both defined on $\mathbb{R}$. Suppose f and g both have vertical tangents at a point a in their domain (i.e., at the same point in the domain). Then, the sum $f + g$ also has a vertical tangent at a.
(7) Suppose f and g are functions, both defined on $\mathbb{R}$. Suppose f and g both have vertical tangents at a point a in their domain (i.e., at the same point in the domain). Then, the pointwise product $f \cdot g$ also has a vertical tangent at a. *This is trickier than it looks!*

3. Max-min problems

Smart thoughts for smart people ...

(1) Before getting started on the messy differentiation to find critical points, think about the constraints and the endpoints. Is it obvious that the function will attain a minimum/maximum at one of the endpoints? What are the values of the function at the endpoints? (If no endpoints, take limiting values as you go in one direction of the domain). Is there an intuitive reason to believe that the function attains its optimal value somewhere *in between* rather than at an endpoint? Is there some kind of trade-off to be made? Are there some things that can be said qualitatively about where the trade-off is likely to occur?
(2) Feel free to convert your function to an equivalent function such that the two functions rise and fall together. This reduces the burden of messy expressions.
(3) *Cobb-Douglas production*: For $p, q > 0$, the function $x \mapsto x^p(1-x)^q$ attains a local maximum at $p/(p+q)$. In fact, this is the absolute maximum on $[0, 1]$, and the function value is $p^p q^q/(p+q)^{p+q}$. This is important because this function appears in disguise all the time (e.g., maximizing area of rectangle with given perimeter, etc.)

(4) A useful idea is that when dividing a resource into two competing uses, and one use is hands-down better than the other, the *best* use happens when the entire resource is devoted to the better use. However, the *worst* may well happen somewhere in between, because divided resources often perform even worse than resources devoted wholeheartedly to a bad use. This is seen in perimeter allocation to boundaries with the objective function being the total area, and area allocation to surfaces with the objective function being the total volume.
(5) When we want to *maximize* something subject to a collection of many constraints, the most relevant constraint is the *minimum* one. Think of the ladder-through-the-hallway problem, or the truck-going-under-bridges problem.

Error-spotting exercises

(1) The absolute maximum among the values of a (?) function (of reals) at integers is attained at the integer closest to the point at which it attains its absolute maximum among all reals.
(2) The absolute maximum among the values of a (?) function (of reals) at integers is attained at one of the integers closest to the point at which it attains a local maximum.
(3) To maximize the sum of two functions is equivalent to maximizing each one separately and then finding the common point of maximum.
(4) If f is a function that is continuous and concave up on an interval $[a, b]$, then the absolute minimum of f always occurs at an interior point and the absolute maximum of f always occurs at an endpoint. *This is a little subtle, because it's almost but not completely correct. Think through it clearly!*
(5) Consider the function:

$$f(x) := \begin{cases} x^3, & 0 \le x \le 1 \\ x^2, & 1 < x \le 2 \end{cases}$$

Then, f' is increasing on $[0, 2]$, so f is concave up on $[0, 2]$.

4. Definite and indefinite integration

4.1. **Definition and basics.** Error-spotting exercises ...

(1) If P_1 and P_2 are partitions of $[a, b]$ and $\|P_2\| \le \|P_1\|$, then P_2 is finer than P_1.
(2) If P_1 and P_2 are partitions of $[a, b]$ such that P_2 is finer than P_1, and f is a bounded function on $[a, b]$, then $L_f(P_2) \le L_f(P_1)$ and $U_f(P_2) \le U_f(P_1)$.
(3) For any continuous function f on $[a, b]$, the number of parts n we need in a regular partition of $[a, b]$ so that the integral is bounded in an interval of length L is proportional to $1/n$.

4.2. **Definite integral, antiderivative, and indefinite integral.** Error-spotting exercises ...

(1) Consider the function $f(x) := \int_x^{x^2} \sin x \, dx$. Then $f'(x) = \sin(x^2) - \sin(x)$.
(2) Suppose f is a function on the nonzero reals such that $f'(x) = 1/x^2$ for all $x \in \mathbb{R}$. Then, we must have $f(x) = 1/x + C$ for some constant C.

4.3. **Higher derivatives, multiple integrals, and initial/boundary conditions.** Error-spotting exercises ...

(1) Suppose F and G are everywhere k times differentiable functions for k a positive integer. If the k^{th} derivatives of the functions F and G are equal, then $F - G$ is a polynomial of degree k.
(2) Suppose F is a function defined on nonzero reals and $F''(x) = 1/x^3$ for all x. Then, F is of the form $F(x) = 1/x + C$ where C is a real constant.

4.4. **Reversing the chain rule.** No error-spotting exercises

4.5. **u-substitutions for definite integrals.** No error-spotting exercises

4.6. **Symmetry and integration.** No error-spotting exercises

4.7. **Mean-value theorem.** No error-spotting exercises

4.8. **Application to area computations.** No error-spotting exercises

5. Graphing and miscellanea on functions

5.1. Symmetry yet again. Words...

(1) All mathematics is the study of symmetry (well, not all).
(2) One interesting kind of symmetry that we often see in the graph of a function is *mirror symmetry* about a vertical line. This means that the graph of the function equals its reflection about the vertical line. If the vertical line is $x = c$ and the function is f, this is equivalent to asserting that $f(x) = f(2c - x)$ for all x in the domain, or equivalently, $f(c + h) = f(c - h)$ whenever $c + h$ is in the domain. In particular, the domain itself must be symmetric about c.
(3) A special case of mirror symmetry is the case of an *even function*. An even function is a function with mirror symmetry about the y-axis. In other words, $f(x) = f(-x)$ for all x in the domain. (Even also implies that the domain should be symmetric about 0).
(4) Another interesting kind of symmetry that we often see in the graph of a function is *half-turn symmetry* about a point on the graph. This means that the graph equals the figure obtained by rotating it by an angle of π about that point. A point (c, d) is a point of half-turn symmetry if $f(x) + f(2c - x) = 2d$ for all x in the domain. In particular, the domain itself must be symmetric about c. If f is defined at c, then $d = f(c)$.
(5) A special case of half-turn symmetry is an odd function, which is a function having half-turn symmetry about the origin.
(6) Another symmetry is *translation symmetry*. A function is *periodic* if there exists $h > 0$ such that $f(x + h) = f(x)$ for all x in the domain of the function (in particular, the domain itself should be invariant under translation by h). If a smallest such h exists, then such an h is termed the period of f.
(7) A related notion is that of a function that is *periodic with shift*. A function is periodic with shift if there exists $h > 0$ and $k \in \mathbb{R}$ such that $f(x + h) - f(x) = k$ for all $x \in \mathbb{R}$. Note that if k is nonzero, the function isn't periodic.

    If f is differentiable for all real numbers, then f' is periodic if and only if f is periodic with shift. In particular, if f' is periodic with period h, then $f(x + h) - f(x)$ is constant. If this constant value is k, then the graph of f has a two-dimensional translational symmetry by (h, k) and its multiples.

    A function that is periodic with shift can be expressed as the sum of a linear function (slope k/h) and a periodic function. The linear part represents the secular trend and the periodic part represents the seasonal variation.

Derivative facts...

(1) The derivative of an even function, if defined everywhere, is odd. Any antiderivative of an odd function is even.
(2) The derivative of an odd function is even. Any antiderivative of an even function is an odd function plus a constant.
(3) The derivative of a function with mirror symmetry has half turn symmetry about the corresponding x-value and has value 0 at that x-value. (For a more detailed description of these, see the solutions to the November 12 whoppers).
(4) Assuming that f' is defined and does not change sign infinitely often on a neighborhood of c, we have that if $x = c$ is an axis of mirror symmetry for the graph of f, then c is a point of local extremum. The reason is that if f is increasing on the immediate left, it must be decreasing on the immediate right, and similarly ...
(5) Assuming that f'' is defined and does not change sign infinitely often on a neighborhood of c, we have that if $(c, f(c))$ is a point of half-turn symmetry for the graph of f, then it is also a point of inflection for the graph. The reason is that if f is concave up on the immediate left, it must be concave down on the immediate right, and similarly ...
(6) The conver statements to the above two do not hold: most points of inflection do not give points of half-turn symmetry, and most local extrema do not give axes of mirror symmetry.
(7) If f has more than one axis of mirror symmetry, then it is periodic. Conversely, if f is periodic with period h, and has an axis of mirror symmetry $x = c$, then all $x = c + (nh/2)$, n an integer, are axes of mirror symmetry.

(8) If f has more than one point of half-turn symmetry, then it is periodic with shift. Conversely, if f is periodic with shift and has a point of half-turn symmetry, it has infinitely many points of half-turn symmetry.

Cute facts...

(1) Constant functions enjoy mirror symmetry about every vertical line and half-turn symmetry about every point on the graph.
(2) Nonconstant linear functions enjoy half-turn symmetry about every point on their graph. They do not enjoy any mirror symmetry (in the sense of mirror symmetry about vertical lines) because they are everywhere increasing or everywhere decreasing. (They do have mirror symmetry about *oblique* lines, but this is not a kind of symmetry that we are considering).
(3) Quadratic (nonlinear) functions enjoy mirror symmetry about the line passing through the vertex (which is the unique absolute maximum/minimum, depending on the sign of the leading coefficient). They do not enjoy any half-turn symmetry.
(4) Cubic functions enjoy half-turn symmetry about the point of inflection, and no mirror symmetry. Either the first derivative does not change sign anywhere, or it becomes zero at exactly one point, or there is exactly one local maximum and one local minimum, symmetric about the point of inflection.
(5) Functions of higher degree do not necessarily have either half-turn symmetry or mirror symmetry.
(6) More generally, we can say the following for sure: a nonconstant polynomial of even degree greater than zero can have at most one line of mirror symmetry and no point of half-turn symmetry. A nonconstant polynomial of odd degree greater than one can have at most one point of half-turn symmetry and no line of mirror symmetry.
(7) The sine function is an example of a function where the points of inflection and the points of half-turn symmetry are the same: the multiples of π. Similarly, the points with vertical axis of symmetry are the same as the points of local extrema: odd multiples of $\pi/2$.
(8) A polynomial is an even function iff all its terms have even degree. Such a polynomial is termed an *even polynomial*. A polynomial is an odd function iff all its terms have odd degree. Such a polynomial is termed an *odd polynomial*.

Actions ...

(1) Worried about periodicity? Don't be worried if you only see polynomials and rational functions. Trigonometric functions should make you alert. Try to fit in the nicest choices of period. Check if smaller periods can work (e.g., for $\sin^2$, the period is π). Even if the function in and of itself is not periodic, it might have a periodic derivative or a periodic second derivative. The sum of a linear function and a periodic function has periodic derivative, and the sum of a quadratic function and a periodic function has a periodic second derivative.
(2) Want to milk periodicity? Use the fact that for a periodic function, the behavior everywhere is just the behavior over one period translates over and over again. If the first derivative is periodic, the increase/decrease behavior is periodic. If the second derivative is periodic, the concave up/down behavior is periodic.
(3) Worried about even and odd, and half-turn symmetry and mirror symmetry? If you are dealing with a quadratic polynomial, or a function constructed largely from a quadratic polynomial, you are probably seeing some kind of mirror symmetry. For cubic polynomials and related constructions, think half-turn symmetry.
(4) Use also the cues about even and odd polynomials.

5.2. **Graphing a function.** Actions ...

(1) To graph a function, a useful first step is finding the domain of the function.
(2) It is useful to find the intercepts and plot a few additional points.
(3) Try to look for symmetry: even, odd, periodic, mirror symmetry, half-turn symmetry, and periodic derivative.
(4) Compute the derivative. Use that to find the critical points, the local extreme values, and the intervals where the function increases and decreases.

(5) Compute the second derivative. Use that to find the points of inflection and the intervals where the function is concave up and concave down.
(6) Look for vertical tangents and vertical cusps. Look for vertical asymptotes and horizontal asymptotes. For this, you may need to compute some limits.
(7) Connect the dots formed by the points of interest. Use the information on increase/decrease and concave up/down to join these points. To make your graph a little better, compute the first derivative (possibly one-sided) at each of these points and start off your graph appropriately at that point.

Subtler points...

(1) When graphing a function, there may be many steps where you need to do some calculations and solve equations and you are unable to carry them out effectively. You can skip some of the steps and come back to them later.
(2) If you cannot solve an equation exactly, try to approximate the locations of roots using the intermediate value theorem or other results such as Rolle's theorem.
(3) In some cases, it is helpful to graph multiple functions together, on the same graph. For instance, we may be interested in graphing a function and its second and higher derivatives. There are other examples, such as graphing a function and its translates, or a function and its multiplicative shifts.
(4) A graph can be used to suggest things about a function that are not obvious otherwise. However, the graph should not be used as conclusive evidence. Rather, the steps used in drawing the graph should be retraced and used to give an algebraic proof.
(5) We are sometimes interested in sketching curves that are not graphs of functions. This can be done by locally expressing the curve piecewise as the graph of a function. Or, we could use many techniques similar to those for graphing functions.
(6) For a function with a piecewise description, we plot each piece within its domain. At the points where the definition changes, determine the one-sided limits of the function and its first and second derivatives. Use this to make the appropriate open circles, asymptotes, etc.

6. Tricky topics

6.1. **Piecewise definition by interval: new issues.** Before looking at these, please review the corresponding material on piecewise definition by interval in the previous midterm review sheet.

(1) Composition involving piecewise definitions is tricky. The limit, continuity and differentiation theorems for composition do not hold for one-sided approach. If one of the functions is decreasing, then things can get flipped. For piecewise definitions, when composing, we need to think clearly about how the intervals transform.

    Please review the midterm question on composition (midterm 1, question 7) of piecewise definitions. The key idea is as follows: for the composition $f \circ g$, we make cases to determine the values of g for which the image under g would land in a particular piece for the definition of f. Considering all cases is extremely painful and we are usually able to take shortcuts based on the nature of the problem.
(2) For a function with piecewise definition, the points where the definition changes are endpoints for each definition, and hence, these points are possible candidates for critical points, points of inflection, and local extreme. They're just *candidates* (so they may not be any of these) but they're worth checking out.
(3) A related helpful concept is that of *how smoothly* a function transitions at a point where its definition changes.
(4) At the one extreme are the discontinuous transitions, where the function has a non-removable discontinuity at the point. Such a transition may be a jump discontinuity (if both one-sided limits are defined but unequal) or something even worse, such as an infinite or oscillatory discontinuity.

    For functions with a discontinuity at a point, it makes sense to talk of one-sided derivatives only from the side where the function is continuous; of course, this one-sided derivative may still not exist.
(5) A somewhat smoother transition occurs where the function is continuous but not differentiable at the point where it changes definition. This is a particular kind of *critical point* for the function definition. Critical points could arise in the form of vertical tangents, vertical cusps, or just plain

points of turning such as for $|x|$ or x^+ at $x = 0$. At such points, it makes sense to try to compute the one-sided derivatives, and these can be computed just by differentiating the piece functions and plugging in at the point. The second derivative does not exist at such points. Also, there is an abrupt change in the nature of concavity at these points.

(6) An even smoother transition occurs if the first derivative is defined at the point. If the first derivative is also defined around the point, then we can start thinking about the second derivative.

(7) More generally, we could think of situations where we want the first k derivatives to be defined at or around the point.

(8) To integrate a function with a piecewise definition, partition the interval of integration in a manner that each part lies within one definition piece. Please review the following two routine problems from Homework 6: Exercise 5.4.55 and 5.4.60. You might want to do a few more suggested problems of the same type.

6.2. **The** $\sin(1/x)$ **examples.**

(1) The $\sin(1/x)$ and related examples are somewhat tricky because the function definition differs at an *isolated point*, namely 0.

(2) To calculate any limit or derivative at a point other than 0, we can do formal computations. However, to calculate the derivative at 0, we *must* use the definition of derivative as a limit of a difference quotient.

(3) For all the facts below, the qualitative conclusions at finite places hold if we replace sin by cos. Those at ∞ change qualitatively.

(4) The function $f_0(x) := \{ \begin{matrix} \sin(1/x), & x \neq 0 \\ 0, & x = 0 \end{matrix}$ is odd and satisfies the intermediate value property but is not continuous at 0. Its limit at $\pm\infty$ is 0, i.e., it has horizontal asymptote the x-axis in both directions.

(5) The function $f_1(x) := \{ \begin{matrix} x\sin(1/x), & x \neq 0 \\ 0, & x = 0 \end{matrix}$ is even and continuous but not differentiable at 0. We can see this from the pinching theorem – it is pinched between $|x|$ and $-|x|$. f_1 is infinitely differentable at all points other than 0. Its limit at $\pm\infty$ is 1, and it approaches this from below in both directions.

(6) The function $f_2(x) := \{ \begin{matrix} x^2\sin(1/x), & x \neq 0 \\ 0, & x = 0 \end{matrix}$ is differentiable at 0, and infinitely differentiable everywhere other than 0, but the derivative is not continuous at 0. The limit $\lim_{x\to 0} f_2'(x)$ does not exist. Note that f_2' is defined everywhere and satisfies the intermediate value property but is not continuous at 0.

f_2 is asymptotic to the line $y = x$ both additively and multiplicatively, as $x \to \pm\infty$.

(7) The function $f_3(x) := \{ \begin{matrix} x^3\sin(1/x), & x \neq 0 \\ 0, & x = 0 \end{matrix}$ is continuously differentiable but not twice differentiable at 0, and infinitely differentiable everywhere other than 0.

f_3 is asymptotic to the line $y = x^2 + C$ as $x \to \pm\infty$, where C is an actual constant (whose value you were supposed to compute in a homework problem).

(8) More generally, consider something such as $p(x)\sin(1/(q(x)))$. This function is not defined at the zeros of q. However, it does not have vertical asymptotes at these points. If a is a root of q and also of p, then the limiting value as $x \to a$ is 0. Otherwise, the limit is undefined but the function oscillates between finite bounds.

In the limit as $x \to \pm\infty$, if the degree of p is less than that of q, the function has horizontal asymptote the y-axis. If their degrees are equal, it has asymptote a finite nonzero value, namely $\lim_{x\to\infty} p(x)/q(x)$. If the degree of p is bigger, it is asymptotic to a polynomial.

For $p(x)\cos(1/(q(x)))$, the behavior at points where the function isn't defined is the same as for sin, but the behavior at $\pm\infty$ is different – the cos part goes to 1, so the function is asymptotically polynomial, albeit not necessarily to p itself.

(9) Fun exercise: Consider $x\tan(1/x)$. What can you say about this?

6.3. **Power functions.** We here consider exponents $r = p/q$, q odd. When q is even, or when r is irrational, the conclusions drawn here continue to hold for $x > 0$; however, the function isn't defined for $x < 0$.

For each of these, you should be able to provide ready justifications/reasoning based on derivatives.

(1) Case $r < 0$: x^r is undefined at 0. It is decreasing and concave up on $(0, \infty)$, with vertical asymptote at $x = 0$ going to $+\infty$ and horizontal asymptote as $x \to \infty$ going to $y = 0$. If p is even, it is increasing and concave up on $(-\infty, 0)$ with horizontal asymptote as $x \to -\infty$ going to $y = 0$ and vertical asymptote $+\infty$ at 0. If p is odd, it is decreasing and concave down on $(-\infty, 0)$ with horizontal asymptote as $x \to -\infty$ going to $y = 0$ and vertical asymptote $-\infty$ at 0.
(2) Case $r = 0$: We get a constant function with value 1.
(3) Case $0 < r < 1$: x^r is increasing and concave down on $(0, \infty)$. If p is even, it is decreasing and concave down on $(-\infty, 0)$ and has a downward-facing vertical cusp at $(0, 0)$. If p is odd, it is increasing and concave up on $(-\infty, 0)$ and has an upward vertical tangent at $(0, 0)$.
(4) Case $r = 1$: A straight line $y = x$.
(5) Case $1 < r$: x^r is increasing and concave up on $(0, \infty)$. If p is even, it is decreasing and concave up on $(-\infty, 0)$ and has a local and absolute minimum and critical point at $(0, 0)$. If p is odd, it is increasing and concave down on $(-\infty, 0)$ and has a point of inflection-type critical point (no local extreme value) at $(0, 0)$.

6.4. **Local behavior heuristics: multiplicative.** You have a complicated looking function such that $(x - \alpha_1)^{r_1}(x - \alpha_2)^{r_2} \dots (x - \alpha_k)^{r_k}$. What is the local behavior of the function near $x = \alpha_1$?

The answer: For determining the qualitative nature of this local behavior, you can just concentrate on $(x - \alpha_i)^{r_i}$ and ignore the rest. In particular, *just* looking at r_1, you can figure out whether you have a critical point, local extremum, point of inflection, vertical tangent, or vertical cusp. The other things *do* matter if you are further interested in, say, whether we have a local maximum or minimum, or in whether the vertical tangent is an increasing tangent or a decreasing tangent, or which direction the vertical cusp points in.

Overall, if $r_i > 0$ and $r_i = p_i/q_i$, q_i odd, and both p_i, q_i positive, then:

- Critical point iff $r_i \neq 1$.
- Local extremum iff p_i even.
- Point of inflection/vertical tangent iff p_i odd.
- Vertical cusp iff p_i even and $p_i < q_i$, both positive, i.e., $r_i < 1$. Note that vertical cusp is a special kind of local extremum.
- Vertical tangent iff p_i odd, $p_i < q_i$, both positive, i.e., $r_i < 1$.

So, if we look at, say $(x - \pi)^2(x - \sqrt{6})^3(x - 2)^{1/3}(x - 3)^{2/3}$, it has a local extremum at π, a point of inflection at $\sqrt{6}$, a vertical tangent at 2, and a vertical cusp at 3.

6.5. **Local behavior heuristics: additive.** If you have something of the form $f + g$, and the vertical tangent/cusp points for f are disjoint from those of g, then the vertical tangent/cusp points for $f + g$ include both lists. Further, the nature (tangent versus cusp) is inherited from the corresponding piece.

For instance, for $x^{1/3} + (x - 131.4)^{2/3}$, there is a vertical tangent at $x = 0$ and a vertical cusp at $x = 131.4$.

In particular, if g is everywhere differentiable, then the vertical tangent/cusp behavior of $f + g$ is the same as that of f.

7. High yield practice

Here are the areas that you should focus on if you have a thorough grasp of the basics:

(1) Everything to do with piecewise definitions (differentiation, integration, reasoning).
(2) Vertical tangents and cusps in sophisticated cases.
(3) Horizontal, oblique, and weird asymptotes.
(4) Trigonometric integrations, particularly $\sin^2$, $\cos^2$, and $\tan^2$ and their variants.
(5) Tricky integration problems that involve the use of symmetry and/or the chain rule.

8. Quickly

This "Quickly" list is a bit of a repeat and augmentation of the "Quicky" list given out for the previous midterm.

8.1. **Arithmetic.** You should be able to:

- Do quick arithmetic involving fractions.
- Remember $\sqrt{2}$, $\sqrt{3}$, and π to at least two digits.
- Sense when an expression will simplify to 0.
- Compute approximate values for square roots of small numbers, π and its multiples, etc., so that you are able to figure out, for instance, whether $\pi/4$ is smaller or bigger than 1, or two integers such that $\sqrt{39}$ is between them.
- Know or quickly compute small powers of small positive integers. This is particularly important for computing definite integrals. For instance, to compute $\int_2^3 (x+1)^3\,dx$, you need to know/compute 3^4 and 4^4.

8.2. **Computational algebra.** You should be able to:

(1) Add, subtract, and multiply polynomials.
(2) Factorize quadratics or determine that the quadratic cannot be factorized.
(3) Factorize a cubic if at least one of its factors is a small and easy-to-spot number such as 0, ± 1, ± 2, ± 3.
(4) Do polynomial long division (not usually necessary, but helpful).
(5) Solve simple inequalities involving polynomial and rational functions once you've obtained them in factored form.

8.3. **Computational trigonometry.** You should be able to:

(1) Determine the values of sin, cos, and tan at multiples of $\pi/2$.
(2) Determine the intervals where sin and cos are positive and negative.
(3) Remember the formulas for $\sin(\pi - x)$ and $\cos(\pi - x)$, as well as formulas for $\sin(-x)$ and $\cos(-x)$.
(4) Recall the values of sin and cos at $\pi/6$, $\pi/4$, and $\pi/3$, as well as at the corresponding obtuse angles.
(5) Reverse lookup for these, for instance, you should quickly identify the acute angle whose sin is $1/2$.

8.4. **Computational limits.** You should be able to: size up a limit, determine whether it is of the form that can be directly evaluated, of the form that we already know does not exist, or indeterminate.

8.5. **Computational differentiation.** You should be able to:

(1) Differentiate a polynomial (written in expanded form) on sight (without rough work).
(2) Differentiate a polynomial (written in expanded form) twice (without rough work).
(3) Differentiate sums of powers of x on sight (without rough work).
(4) Differentiate rational functions with a little thought.
(5) Do multiple differentiations of expressions whose derivative cycle is periodic, e.g., $a \sin x + b \cos x$.
(6) Differentiate simple composites without rough work (e.g., $\sin(x^3)$).

8.6. **Computational integration.** You should be able to:

(1) Compute the indefinite integral of a polynomial (written in expanded form) on sight without rough work.
(2) Compute the definite integral of a polynomial with very few terms within manageable limits quickly.
(3) Compute the indefinite integral of a sum of power functions quickly.
(4) Know that the integral of sine or cosine on any quadrant is ± 1.
(5) Compute the integral of $x \mapsto f(mx)$ if you know how to integrate f. In particular, integrate things like $(a + bx)^m$.
(6) Integrate sin, cos, $\sin^2$, $\cos^2$, $\tan^2$, $\sec^2$, $\cot^2$, $\csc^2$,.

8.7. **Being observant.** You should be able to look at a function and:

(1) Sense if it is odd (even if nobody pointedly asks you whether it is).
(2) Sense if it is even (even if nobody asks you whether it is).
(3) Sense if it is periodic and find the period (even if nobody asks you about the period).

8.8. **Graphing.** You should be able to:

(1) Mentally graph a linear function.
(2) Mentally graph a power function x^r (see the list of things to remember about power functions). Sample cases for r: $1/3$, $2/3$, $4/3$, $5/3$, $1/2$, 1, 2, 3, -1, $-1/3$ $-2/3$.
(3) Graph a piecewise linear function with some thought.
(4) Mentally graph a quadratic function (very approximately) – figure out conditions under which it crosses the axis etc.
(5) Graph a cubic function after ascertaining which of the cases for the cubic it falls under.
(6) Mentally graph sin and cos, as well as functions of the $A\sin(mx)$ and $A\cos(mx)$.
(7) Graph a function of the form linear + trigonometric, after doing some quick checking on the derivative.

8.9. **Fancy pictures.** Keep in mind approximate features of the graphs of:

(1) $\sin(1/x)$, $x\sin(1/x)$, $x^2\sin(1/x)$ and $x^3\sin(1/x)$, and the corresponding cos counterparts – both the behavior near 0 and the behavior near $\pm\infty$.
(2) The Dirichlet function and its variants – functions defined differently for the rationals and irrationals.