REVIEW SHEET FOR MIDTERM 1

MATH 152, SECTION 55 (VIPUL NAIK)

The document is arranged as follows. The initial sections/subsections correspond to topics. Each subsection has two sets of points, "Words" which includes basic theory and definitions, and "Actions" which provides information on strategies for specific problem types. In some cases, there are additional points. The lists of points are largely the same as the executive summaries at the beginning of the lecture notes, though some additional points (that make sense now but wouldn't have made sense at the time the lecture was delivered) have been added.

For each subsection, there is also an "Error-spotting exercises" list. We will be doing these exercises in the review session, though you may benefit by trying them out in advance. For the simpler topics, we may do *only* the error-spotting exercises in the review session so as to save time and concentrate on the harder topics.

The section titled "Tricky topics" covers a bunch of topics and question types that habitually confuse students. This includes piecewise definitions by interval, piecewise definition by rational-irrational, $\sin(1/x)$ examples, and thinking about counterexamples to statements. You need to read each point carefully and then try to locate examples from class, homeworks, or quiz questions.

The section titled "High yield practice" lists (without details) the areas where I think practice is most helpful if you feel you're already fairly thorough with the basic formulas. If you feel you are on top of AP-level material, for instance, then these are the areas where most of your energies should be devoted.

The end of the document has some "Quickly" lists. These are lists of things you should be able to accomplish quickly. This includes numerical values, formulas, graphs, examples, and counterexamples, that should be ready for immediate recall in the test environment. I simply provide a list and do not include details of all the formulas and graphs.

To maximize efficiency in the review session, here is what I suggest. Go through all the lists of points. For each point, make sure you understand it by jotting down a relevant example or illustration or providing a brief justification. If you have difficulty, go back to the lecture notes and read them in detail. You might also want to look at more worked examples in the book, and check out homework and quiz problems.

If everybody is on top of the basic material, we will go very quickly over Sections 1–4 and Section 7 and concentrate most of our energies on Section 5 ("Tricky topics") and Section 6 ("High yield practice").

1. Functions

1.1. **Review part 1.** Words ...

(1) The *domain* of a function is the set of possible inputs. The *range* is the set of possible outputs. When we say $f : A \to B$ is a function, we mean that the domain is A, and the range is a *subset* of B (possibly equal to B, but also possibly a proper subset).
(2) The main fact about functions is that *equal inputs give equal outputs*. We deal here with functions whose domain and range are both subsets of the real numbers.
(3) We typically define a function using an algebraic expression, e.g. $f(x) := 3+\sin x$. When an algebraic expression is given without a specified domain, we take the domain to be the largest possible subset of the real numbers for which the function makes sense.
(4) Functions can be defined piecewise, i.e., one definition on one part of the domain, another definition on another part of the domain. Interesting things happen where the function changes definition.
(5) Functions involving absolute values, max of two functions, min of two functions, and other similar constructions end up having piecewise definitions.

Actions (think back to examples where you've dealt with these issues)...

(1) To find the (maximum possible) domain of a function given using an expression, exclude points where:
   (a) Any denominator is zero.
   (b) Any expression under the square root sign is negative.
   (c) Any expression under the square root sign in the denominator is zero or negative.
(2) To find whether a given number a is in the range of a function f, try solving $f(x) = a$ for x in the domain.
(3) To find the range of a given function f, try solving $f(x) = a$ with a now being an *unknown constant*. Basically, solve for x in terms of a. The set of a for which there exists one or more value of x solving the equation is the range.
(4) To write a function defined as $H(x) := \max\{f(x), g(x)\}$ or $h(x) := \min\{f(x), g(x)\}$ using a piecewise definition, find the points where $f(x) - g(x)$ is zero, find the points where it is positive, and find the points where it is inegative. Accordingly, define h and H on those regions as f or g. *Added: Note that when f and g are both continuous everywhere, then the functions can cross each other only at points where they become equal. However, if the two functions are not everywhere continuous, the functions can cross each other at points of discontinuity as well.*
(5) To write a function defined as $h(x) := |f(x)|$ piecewise, split into regions based on the sign of $f(x)$.
(6) To solve an equation for a function with a piecewise definition, solve for each definition within the piece (domain) for which that definition is satisfied.

Error-spotting exercises ...

Warning: There may be one or more errors in each item.

(1) Consider the function $f(x) := \sqrt{x-1} + \sqrt{2-x}$. The domain of $\sqrt{x-1}$ is $[1, \infty)$ and the domain of $\sqrt{2-x}$ is $(-\infty, 2]$. The domain of the sum is therefore the union of the domains, which is $(-\infty, \infty)$, i.e., the set of all real numbers.
(2) Consider the function $f(x) := \sqrt{(x-1)(x-2)}$. This is the product of the functions $x \mapsto \sqrt{x-1}$ and $x \mapsto \sqrt{x-2}$, hence its domain is the intersection of the domains of the two functions, which are $[1, \infty)$ and $[2, \infty)$ respectively. The domain of f is thus $[2, \infty)$.
(3) Consider the function $f(x) := \max\{x-1, 2x+1\}$. Then, we get $(f(x))^2 = \max\{(x-1)^2, (2x+1)^2\}$.

1.2. **Review part 2.** Note: Although the lecture notes (and the executive summary in front of them) cover the notion of miror symmetry and half-turn symmetry in greater generality than just looking at even and odd functions, I didn't get time to cover this in class. Since this is also not in the book, we will omit these more general notions of symmetry for this midterm. We'll probably cover them when we turn to graphing functions.

Words ...

(1) Given two functions f and g, we can define pointwise combinations of f and g: the sum $f + g$, the difference $f - g$, the product $f \cdot g$, and the quotient f/g. For the sum, difference, and product, the domain is the intersection of the domains of f and g. For the quotient, the domain is the intersection of the domain of f and the set of points where g takes a nonzero value.
(2) Given a function f and a real number α, we can consider the scalar multiple αf.
(3) Given two functions f and g, we can try talking of the composite function $f \circ g$. This is defined for those points in the domain of g whose image lies in the domain of f.
(4) An *even function* is a function with mirror symmetry about the y-axis. In other words, $f(x) = f(-x)$ for all x in the domain. (Even also implies that the domain should be symmetric about 0).
(5) An odd function is a function having half-turn symmetry about the origin. By definition, the domain of an odd function is symmetric about $\mathbb{R}$. An odd function, if defined at 0, takes the value 0 at 0.
(6) A function f defined on $\mathbb{R}$ is periodic if there exists $h > 0$ such that $f(x+h) = f(x)$ for every $x \in \mathbb{R}$. If there is a smallest $h > 0$ satisfying this, such a h is termed the *period*. Constant functions are periodic but have no period. The sine and cosine functions are periodic with period 2π.

Actions ...

(1) To prove that a function is periodic, try to find a h that *works* for every x. To prove that a function is periodic but has no period, try to show that there are arbitrarily small $h > 0$ that work.

(2) To prove that a function is even or odd, just try proving the corresponding equation for all x. Nothing but algebra.
(3) If a function is defined for the positive or nonnegative reals and you want to extend the definition to negatives to make it even or odd, extend it so that the formula is preserved. So define $f(-x) = f(x)$, for instance, to make it even.

Error-spotting exercises ...

(1) We know that odd + odd = even, so the sum of two odd functions is an even function.
(2) Consider the function $\sin^2 x = \sin(\sin x)$. Since the period of the sin function is 2π, the period of the $\sin^2$ function is also 2π.
(3) Suppose f is a periodic function with period h_f and g is a periodic function with period h_g. Then, the period of $f + g$ is $h_f + h_g$.
(4) Suppose f is a periodic function with period h_f and g is a periodic function with period h_g. Then, the period of the composite $f \circ g$ is $h_f h_g$.
(5) The period of the function $x \mapsto \sin(x^2)$ is $\sqrt{\pi}$.

2. Limits

2.1. **Informal introduction to limits.** Words ...

(1) On the real line, there are two directions from which to approach a point: the *left* direction and the *right* direction.
(2) For a function f, $\lim_{x\to c} f(x)$ is read as "the limit as x approaches c of $f(x)$. Equivalently, as x approaches c, $\lim_{x\to c} f(x)$ is the value that $f(x)$ approaches.
(3) $\lim_{x\to c} f(x)$ makes sense only if f is defined *around* c, i.e., both to the immediate left and to the immediate right of c.
(4) We have the notion of the *left hand limit* $\lim_{x\to c^-} f(x)$ and the *right hand limit* $\lim_{x\to c^+} f(x)$. The *limit* $\lim_{x\to c} f(x)$ exists if and only if (both the left hand limit and the right hand limit exist and they are both equal).
(5) f is termed *continuous* at c if c is in the domain of f, the limit of f at c exists, and $f(c)$ equals the limit. f is termed *left continuous* at c if the left hand limit exists and equals $f(c)$. f is termed *right continuous* at c if the right hand limit exists and equals $f(c)$.
(6) f is termed *continuous* on an interval I in its domain if f is continuous at all points in the interior of I, continuous from the right at any left endpoint in I (if I is closed from the left) and continuous fromthe left at any right endpoint in I (if I is closed from the right).
(7) A *removable discontinuity* for f is a discontinuity where a two-sided limit exists but is not equal to the value. A *jump discontinuity* is a discontinuity where both the left hand limit and right hand limit exist but they are not equal.

Error-spotting exercises...

(1) Consider the function $f(x) := 1/x$. At $x = 0$, both the left hand limit and the right hand limit are equal to each other (since they both do not exist), so f has a limit at $x = 0$.
(2) If f and g both have removable discontinuities at $x = c$, then $f + g$ also has a removable discontinuity at $x = c$.
(3) If f and$g$ both have jump discontinuities at $x = c$, then $f + g$ also has a jump discontinuity at $x = c$.

2.2. **Formal definition of limits.** Words ...

(1) $\lim_{x\to c} f(x) = L$ if, for every $\epsilon > 0$, there exists $\delta > 0$ such that, for every $x \in \mathbb{R}$ satisfying $0 < |x - c| < \delta$ (in other words, $x \in (c - \delta, c) \cup (c, c + \delta)$, we have $|f(x) - L| < \epsilon$ (in other words, $f(x) \in (L - \epsilon, L + \epsilon)$.
(2) What that means is that however small a trap (namely ϵ) the skeptic demands, the person who wants to claim that the limit does exist can find a δ such that when the x-value is δ-close to c, the $f(x)$-value is ϵ-close to L.
(3) The negation of the statement $\lim_{x\to c} f(x) = L$ is: there exists $\epsilon > 0$ such that for every $\delta > 0$ there exists $x \in \mathbb{R}$ such that $0 < |x - c| < \delta$ but $|f(x) - L| \geq \epsilon$.

(4) The statement $\lim_{x \to c} f(x)$ doesn't exist: for every $L \in \mathbb{R}$, there exists $\epsilon > 0$ such that for every $\delta > 0$ there exists $x \in \mathbb{R}$ such that $0 < |x - c| < \delta$ but $|f(x) - L| \geq \epsilon$.
(5) We can think of $\epsilon - \delta$ limits as a game. The skeptic, who is unconvinced that the limit is L, throws to the prover a value $\epsilon > 0$. The prover must now throw back a $\delta > 0$ that works. L being the limit means that the prover has a winning strategy, i.e., the prover has a wayof picking, for any $\epsilon > 0$, a value of $\delta > 0$ suitable to that ϵ.
(6) The function $f(x) = \sin(1/x)$ is a classy example of a limit not existing. The problem is that, however small we choose a δ around 0, the function takes all values between -1 and 1, and hence refuses to be confined within small ϵ-traps.
(7) We say that f is continuous at c if $\lim_{x \to c} f(x) = f(c)$.

Actions...

(1) If a δ works for a given ϵ, then every smaller δ works too. Also, if a δ works for a given ϵ, the same δ works for any larger ϵ.
(2) Constant functions are continuous, we can choose δ to be anything. In this $\epsilon - \delta$ game, the person trying to prove that the limit does exist wins no matter what ϵ the skeptic throws and no matter what δ is thrown back.
(3) For the function $f(x) = x$, it's continuous, and $\delta = \epsilon$ works.
(4) For a linear function $f(x) = ax + b$ with $a \neq 0$, it's continuous, and $\delta = \epsilon/|a|$ works. That's the largest δ that works.
(5) For a function $f(x) = x^2$ taking the limit at a point p, the limit is p^2 (the function is continuous) and $\delta = \min\{1, \epsilon/(1 + |2p|)\}$ works. It isn't the best, but it works.
(6) For a function $f(x) = ax^2 + bx + c$, taking the limit at a point p, the limit is $f(p)$ (the function is continuous) and $\delta = \min\{1, \epsilon/(|a| + |2ap + b|)\}$ works. It isn't the best, but it works.
(7) If there are two functions f and g and $\lim_{x \to c} f(x) = \lim_{x \to c} g(x) = L$, and h is a function such that $h(x) = f(x)$ or $h(x) = g(x)$ for every x, then $\lim_{x \to c} h(x) = L$. The δ that works for h is the minimum of the δs that work for f and g. This applies to many situations: functions defined differently on the left and right of the point, functions defined differently for the rationals and the irrationals, functions defined as the max or min of two functions.

Error-spotting exercises ...

(1) Consider the limit game for $\lim_{x \to c} f(x) = L$. This is a game between the prover and the skeptic. In this game, if the prover wins, then the limit statement is true. If the skeptic wins, then we say that the limit statement is false.
(2) A winning strategy for the prover in the game for $\lim_{x \to c} f(x) = L$ involves the prover fooling the skeptic choosing a suitably large value of ϵ so that the prover can trap the function appropriately.
(3) Consider two continuous functions f and g on the reals with $f(c) = g(c) = L$. Consider $h(x) := \min\{f(x), g(x)\}$ and $H(x) := \max\{f(x), g(x)\}$. The winning strategy for the prover for showing that $\lim_{x \to c} h(x) = L$ is to pick, for any given ϵ, the *minimum* of the δs that work for f and for g. The winning strategy for H is to pick, for any given ϵ, the *maximum* of the δs that work for f and for g.

2.3. **Limit theorems + quick/intuitive calculation of limits.** Words...

(1) If the limits for two functions exist at a particular point, the limit of the sum exists and equals the sum of the limits. Similarly for product and difference.
(2) For quotient, we need to add the caveat that the limit of the denominator is nonzero.
(3) If $\lim_{x \to c} f(x) = L \neq 0$ and $\lim_{x \to c} g(x) = 0$, then $\lim_{x \to c}(f(x)/g(x))$ is undefined.
(4) If $\lim_{x \to c} f(x) = \lim_{x \to c} g(x) = 0$, then we cannot say anything offhand about $\lim_{x \to c}(f(x)/g(x))$.
(5) Everything we said (or implied) can be reformulated for one-sided limits.

Error-spotting exercises...

(1) Suppose f and g are functions both defined around a point c. If $\lim_{x \to c} f(x)$ does not exist and $\lim_{x \to c} g(x)$ does not exist, then $\lim_{x \to c}(f(x) + g(x))$ does not exist either.

2.4. **Continuity theorems.** Words ...

(1) If f and g are functions that are both continuous at a point c, then the function $f+g$ is also continuous at c. Similarly, $f-g$ and $f \cdot g$ are continuous at c. Also, if $g(c) \neq 0$, then f/g is continuous at c.
(2) If f and g are both continuous in an interval, then $f+g$, $f-g$ and $f \cdot g$ are also continuous on the interval. Similarly for f/g provided g is not zero anywhere on the interval.
(3) The composition theorem for continuous functions states that if g is continuous at c and f is continuous at $g(c)$, then $f \circ g$ is continuous at c. The corresponding composition theorem for limits is *not true but almost true*: if $\lim_{x \to c} g(x) = L$ and $\lim_{x \to L} f(x) = M$, then $\lim_{x \to c} f(g(x)) = M$.
(4) The one-sided analogues of the theorems for sum, difference, product, quotient work, but the one-sided analogue of the theorem for composition is not in general true.
(5) Each of these theorems at points has a suitable analogue/corollary for continuity (and, with the exception of composition, for one-sided conitnuity) on intervals.

Error-spotting exercises ...

(1) Suppose f and g are both functions defined and continuous around a point $c \in \mathbb{R}$. Then, $f+g$, $f-g$, $f \cdot g$, and $f \circ g$ are all defined and continuous around c.
(2) Suppose f and g are both continuous functions on the domain $[0,1]$. Then, $f+g$, $f-g$, $f \cdot g$, and $f \circ g$ are all continuous functions on the domain $[0,1]$.
(3) Suppose f and g are both left continuous functions on the domain $[0,1]$. Then, $f+g$, $f-g$, $f \cdot g$, and $f \circ g$ are all left continuous functions on the domain $[0,1]$.
(4) Suppose $\lim_{x \to 0} g(x)/x = A \neq 0$. Then, we have:

$$\lim_{x \to 0} \frac{g(g(x))}{x} = \lim_{x \to 0} \frac{g(g(x))}{g(x)} \lim_{x \to 0} \frac{g(x)}{x} = g\left(\lim_{x \to 0} \frac{g(x)}{x}\right) \cdot \lim_{x \to 0} \frac{g(x)}{x} = g(A) \cdot A = Ag(A)$$

(5) Suppose $\lim_{x \to 0} g(x)/x^2 = A \neq 0$. Then, we have:

$$\lim_{x \to 0} \frac{g(g(x))}{x^4} = \lim_{x \to 0} \frac{g(g(x))}{(g(x))^2} \lim_{x \to 0} \frac{g(x)}{x^2} = A \cdot A = A^2$$

2.5. **Three important theorems.** Words ...

(1) The pinching theorem states that if $f(x) \leq g(x) \leq h(x)$, and $\lim_{x \to c} f(x) = \lim_{x \to c} h(x) = L$, then $\lim_{x \to c} g(x) = L$. A one-sided version of the pinching theorem also holds.
(2) The intermediate-value theorem states that if f is a continuous function, and $a < b$, and p is between $f(a)$ and $f(b)$, there exists $c \in [a,b]$ such that $f(c) = p$. Note that we need f to be defined and continuous on the entire closed interval $[a,b]$.
(3) The extreme-value theorem states that on a closed bounded interval $[a,b]$, a continuous function attains its maximum and minimum.

Actions ...

(1) When trying to calculate a limit that's tricky, you might want to bound it from both sides by things whose limits you know and are equal. For instance, the function $x \sin(1/x)$ taking the limit at 0, or the function that's x on rationals and 0 on irrationals, again taking the limit at 0.
(2) We can use the intermediate-value theorem to show that a given equation has a solution in an interval by calculating the values of the expression at endpoints of the interval and showing that they have opposite signs.

Error-spotting exercises ...

(1) Consider the function $f(x) := 1/x$ on the interval $[-1,1]$. We have $f(-1) = -1$ and $f(1) = 1$, so by the intermediate value theorem, there exists $x \in [-1,1]$, such that $f(x) = 1/2$. Thus, we get that $1/x = 1/2$, so $x = 2$ is in the interval $[-1,1]$.
(2) By the intermediate value theorem, the image of a closed interval $[a,b]$ under a continuous function f is the closed interval $[f(a), f(b)]$ if $f(a) < f(b)$ and the closed interval $[f(b), f(a)]$ if $f(b) < f(a)$.

3. Derivatives

3.1. Derivatives: basics. Words ...

1. For a function f, we define the *difference quotient* between w and x as the quotient $(f(w)-f(x))/(w-x)$. It is also the slope of the line joining $(x, f(x))$ and $(w, f(w))$. This line is called a *secant line*. The segment of the line between the points x and w is sometimes termed a *chord*.
2. The limit of the difference quotient is defined as the *derivative*. This is the slope of the *tangent line* through that point. In other words, we define $f'(x) := \lim_{w \to x} \frac{f(w)-f(x)}{w-x}$. This can also be defined as $\lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$.
3. If the derivative of f at a point x exists, the function is termed *differentiable* at x.
4. If the derivative at a point exists, then the tangent line to the graph of the function exists and its slope equals the derivative. The tangent line is horizontal if the derivative is zero. Note that if the derivative exists, then the tangent line cannot be vertical.
5. Here are some misconceptions about tangent lines: (i) that the tangent line is the line perpendicular to the radius (this makes sense only for circles) (ii) that the tangent line does not intersect the curve at any other point (this is true for some curves but not for others) (iii) that any line other than the tangent line intersects the curve at at least one more point (this is always false – the vertical line through the point does not intersect the curve elsewhere, but is not the tangent line if the function is differentiable).
6. In the Leibniz notation, if y is functionally dependent on x, then $\Delta y/\Delta x$ is the difference quotient – it is the quotient of the difference between the y-values corresponding to x-values. The limit of this, which is the derivative, is dy/dx.
7. The left-hand derivative of f is defined as the left-hand limit for the derivative expression. It is $\lim_{h \to 0^-} \frac{f(x+h)-f(x)}{h}$. The right-hand derivative is $\lim_{h \to 0^+} \frac{f(x+h)-f(x)}{h}$.
8. Higher derivatives are obtained by differentiating again and again. The *second* derivative is the derivative of the derivative. The n^{th} derivative is the function obtained by differentiating n times. In prime notation, the second derivative is denoted f'', the third derivative f''', and the n^{th} derivative for large n as $f^{(n)}$. In the Leibniz notation, the n^{th} derivative of y with respect to x is denoted $d^n y/dx^n$.
9. Derivative of sum equals sum of derivatives. Derivative of difference is difference of derivatives. Scalar multiples can be pulled out.
10. We have the *product rule* for differentiating products: $(f \cdot g)' = f' \cdot g + f \cdot g'$.
11. We have the *quotient rule* for differentiating quotients: $(f/g)' = (g \cdot f' - f \cdot g')/g^2$.
12. The derivative of x^n with respect to x is nx^{n-1}.
13. The derivative of sin is cos and the derivative of cos is $-\sin$.
14. The chain rule says that $(f \circ g)' = (f' \circ g) \cdot g'$

Actions ...

1. We can differentiate any polynomial function of x, or a sum of powers (possibly negative powers or fractional powers), by differentiating each power with respect to x.
2. We can differentiate any rational function using the quotient rule and our knowledge of how to differentiate polynomials.
3. We can find the equation of the tangent line at a point by first finding the derivative, which is the slope, and then finding the point's coordinates (which requires evaluating the function) and then using the point-slope form.
4. Suppose g and h are everywhere differentiable functions. Suppose f is a function that is g to the left of a point a and h to the right of the point a, and suppose $f(a) = g(a) = h(a)$. Then, the left-hand derivative of f at a is $g'(a)$ and the right-hand derivative of f at a is $h'(a)$.
5. The k^{th} derivative of a polynomial of degree n is a polynomial of degree $n-k$, if $k \leq n$, and is zero if $k > n$.
6. We can often use the sum rule, product rule, etc. to find the values of derivatives of functions constructed from other functions simply using the values of the functions and their derivatives at

specific points. For instance, $(f \cdot g)'$ at a specific point c can be determined by knowing $f(c)$, $g(c)$, $f'(c)$, and $g'(c)$.

(7) Given a function f with some unknown constants in it (so a function that is not completely known) we can use information about the value of the function and its derivatives at specific points to determine those constant parameters.

Error-spotting exercises ...

(1) The derivative of the function $x \mapsto \sin^2 x$ is:

$$\frac{d}{dx}(\sin(x^2)) = -\cos(x^2)$$

because the derivative of sin is $-\cos$.

(2) The derivative of the function $x \mapsto \sin(x^2)$ is:

$$\frac{d}{dx}(\sin(x^2)) = (\cos x)(2x) = 2x \cos x$$

(3) The derivative of the function $x \mapsto x \cos x$ is:

$$\frac{d}{dx}(x \cos x) = \frac{d}{dx}(x) \cdot \frac{d}{dx}(\cos x) = (1)(-\sin x) = -\sin x$$

3.2. **Tangents and normals: geometry.** Words...

(1) The normal line to a curve at a point is the line perpendicular to the tangent line. Since the tangent line is the best linear approximation to the curve at the point, the normal line can be thought of as the line best approximating the perpendicular line to the curve.
(2) The angle of intersection between two curves at a point of intersection is defined as the angle between the tangent lines to the curves at that point. If the slopes of the tangent lines are m_1 and m_2, the angle is $\pi/2$ if $m_1 m_2 = -1$. Otherwise, it is the angle α such that $\tan \alpha = |m_1 - m_2|/(|1 + m_1 m_2|)$.
(3) If the angle between two curves at a point of intersection is $\pi/2$, they are termed *orthogonal* at that point. If the curves are orthogonal at all points of intersection, they are termed *orthogonal curves*.
(4) If the angle between two curves at a point of intersection is 0, that means they have the same tangent line. In this case, we say that the curves *touch* each other or are *tangent* to each other.

Actions...

(1) The equation of the normal line to the graph of a function f at the point $(x_0, f(x_0))$ is $f'(x_0)(y - f(x_0)) + (x - x_0) = 0$. The slope is $-1/f'(x_0)$.
(2) To find the angle(s) of intersection between two curves, we first find the point(s) of intersection, then compute the value of derivative (or slope of tangent line) to both curves, and then finally plug that in the formula for the angle of intersection.
(3) It is also possible to find all tangents to a given curve, or all normals to a given curve, that pass through a given point *not* on the curve. To do this, we set up the generic expression for a tangent line or normal line to the curve, and then plug into that generic expression the specific coordinates of the point, and solve. For instance, the generic equation for the tangent line to the graph of a function f is $y - f(x_1) = f'(x_1)(x - x_1)$ where $(x_1, f(x_1))$ is the point of tangency. Plugging in the point (x, y) that we know the curve passes through, we can solve for x_1.
(4) In many cases, it is possible to determine geometrically the number of tangents/normals passing through a point outside the curve. Also, in some cases, the algebraic equations may not be directly solvable, but we may be able to determine the number and approximate location of the solutions.

3.3. **Deeper perspectives on derivatives.** Words...

(1) A continuous function that is everywhere differentiable need not be everywhere continuously differentiable.
(2) If f and g are functions that are both continuously differentiable (i.e., they are differentiable and their derivatives are continuous functions), then $f + g$, $f - g$, $f \cdot g$, and $f \circ g$ are all continuously differentiable.

(3) If f and g are functions that are both k times differentiable (i.e., the k^{th} derivatives of the functions f and g exist), then $f+g$, $f-g$, $f \cdot g$, and $f \circ g$ are also k times differentiable.
(4) If f and g are functions that are both k times continuously differentiable (i.e., the k^{th} derivatives of both functions exist and are continuous) then $f+g$, $f-g$, and $f \cdot g$, and $f \circ g$ are also k times continuously differentiable.
(5) If f is k times differentiable, for $k \geq 2$, then it is $k-1$ times continuously differentiable, i.e., the $(k-1)^{th}$ derivative of f is a continuous function.
(6) If a function is *infinitely differentiable*, i.e., it has k^{th} derivatives for all k, then its k^{th} derivatives are continuous functions for all k.

Error-spotting exercises...

(1) Suppose f and g are everywhere defined functions such that f is twice differentiable and g is three times differentiable. Then the function $f+g$ is $2+3=5$ times differentiability and the function $f \cdot g$ is $2 \cdot 3 = 6$ times differentiable.
(2) A polynomial function of degree n is n times differentiable but not $n+1$ times differentiable, because the $(n+1)^{th}$ and higher derivatives all vanish.
(3) The function $x \mapsto x^{11/3}$ is infinitely differentiable everywhere because we can keep applying the differentiation formula as many times as we want.

4. Trigonometry: review, limits, and derivatives

(1) The following three important limits form the foundation of trigonometric limits: $\lim_{x \to 0} (\sin x)/x = 1$, $\lim_{x \to 0} (\tan x)/x = 1$, and $\lim_{x \to 0} (1 - \cos x)/x^2 = 1/2$.
(2) The derivative of sin is cos, the derivative of cos is $-\sin$. The derivative of tan is $\sec^2$, the derivative of cot is $-\csc^2$, the derivative of sec is $\sec \cdot \tan$, and the derivative of csc is $-\csc \cdot \cot$.
(3) The second derivative of any function of the form $x \mapsto a \sin x + b \cos x$ is the negative of that function, and the fourth derivative is the original function.

Actions ...

(1) Substitution is one trick that we use for trigonometric limits: we translate $\lim_{x \to c}$ to $\lim_{h \to 0}$ where $x = c + h$.
(2) Multiplicative splitting, chaining, and stripping are some further tricks that we often use.
(3) For derivatives of functions that involve composites of trigonometric and polynomial functions, we *have* to use the chain rule as well as rules for sums, differences, products, and quotients when simplifying expressions.

5. Tricky topics

These are some tricky question types and some stumbling blocks across multiple question types. The selection here is based on class feedback, your quiz scores, and concerns raised in problem session.

Some of this is a repeat of points given earlier. However, it may be helpful to have the information presented in this alternative format.

5.1. **Piecewise definition by interval: left and right.** *Think of examples* for each point. I've deliberately not included examples, because I want you to puzzle out each point here in terms of things like this that you've seen. The material straddles limits, continuity, and differentiability.

(1) It is often the case that we define a function piecewise by splitting the domain into intervals. Here, a function has different expressions defining it on different intervals. For the remaining observations, we will assume that each of the piece functions itself is very nice (continuous, differentiable, etc.) so that most of the trouble arises from the changes in definition between intervals.
(2) At a point that is at the common boundary of two intervals, the function changes definition. The function at the boundary point may be defined using either of the two intervals, or separately, as an isolated definition just at that point. (Think of examples).
(3) If the point is included in the definition on one side, it is automatically continuous from that side. (Remember, we're assuming that the piece functions are continuous). For the other side, we need to

calculate the appropriate one-sided limit. If that piece function extends continuously to the point, we substitute the value. Otherwise, we use the limits techniques. (Think of examples).

(4) If the function is continuous from a particular side, that one-sided derivative can be calculated by differentiating the expression formally at the point and evaluating at the point. (Think of examples).

(5) The function is differentiable at a point of definition change if: (i) it is continuous (from both sides) and (ii) the left hand derivative and the right hand derivative agree.

(6) To calculate second or higher derivatives of functions with piecewise definitions, first get a piecewise definition for the function and then differentiate it.

(7) To do an $\epsilon - \delta$ proof of continuity for a function at a point where it may be changing definition, we need to find a δ that works for each piece, and then pick the minimum of those δs.

(8) In order to add, subtract, or multiply two functions with piecewise definitions, we need to break the domains into further pieces so that the pieces for both functions match up. (In mathematical jargon, this is a common refinment). Then we can add, subtract, and multiply in each piece.

(9) Also note that the limit, continuity, and differentiation formulas hold for one-sided approach.

(10) Composition involving piecewise definitions is tricky. The limit, continuity and differentiation theorems for composition do not hold for one-sided approach. If one of the functions is decreasing, then things can get flipped. For piecewise definitions, when composing, we need to think clearly about how the intervals transform.

Error-spotting exercises ...

(1) Consider the function:

$$f(x) := \begin{cases} 1, & x = 0 \\ x + 1, & x > 0 \\ x + \cos x, & x < 0 \end{cases}$$

The derivative is given as follows:

$$f'(x) := \begin{cases} 0, & x = 0 \\ 1, & x > 0 \\ 1 - \sin x, & x < 0 \end{cases}$$

(2) Consider the function:

$$f(x) := \begin{cases} x^2, & x < 0 \\ x^3, & x \geq 0 \end{cases}$$

Then the function $f(x) + f(1-x)$ is:

$$f(x) + f(1-x) = \begin{cases} x^2 + (1-x)^2, & x < 0 \\ x^3 + (1-x)^3, & x \geq 0 \end{cases}$$

5.2. **Piecewise definitions: rational and irrational.**

(1) Sometimes, we may define a function f as one thing for rational inputs and another thing for irrational inputs. The important thing to remember is that *every open interval contains both rational and irrational numbers*. Hence, however small an interval we choose, both definitions are operational in that interval. This is in sharp contrast to the piecewise definition by interval, where different definitions operate in different regions. We'll assume that both piece definitions are obtained by restriction from continuous functions on $\mathbb{R}$.

(2) A function f defined this way is continuous if both the rational and irrational definitions "agree" at the point. (This is assuming that both piece definitions are drawn from continuous functions of $\mathbb{R}$).

(3) An $\epsilon - \delta$ proof of continuity would find the δ that works for the rational and irrational pieces and use the minimum of these. The proof would involve splitting into cases for x based on whether x is rational or irrational.

(4) If both piece definitions are drawn from differentiable functions on $\mathbb{R}$, then f is differentiable at a point if the rational and irrational definitions for both f and f' "agree" at the point.

(5) In order for a second or higher derivative to exist, the first derivative must be defined in a neighborhood of the point. Note that in this sense, the rational-irrational version differs radically from the left-right version. For instance, consider:

$$f(x) := \{ \begin{array}{ll} x^3, & x \leq 0 \\ x^5, & x > 0 \end{array}$$

and

$$g(x) := \{ \begin{array}{ll} x^3, & x \text{ rational} \\ x^5, & x \text{ irrational} \end{array}$$

For both f and g, the function is continuous, and both "piece" derivatives at 0 are 0, so f and g are both differentiable at 0 and $f'(0) = g'(0) = 0$.

However, the situation becomes different with the second derivative. It turns out that $f''(0)$ exists and equals 0. But we cannot talk of $g''(0)$, because, *although $g'(0)$ exists, g' is not defined anywhere around 0, so it does not make sense to differentiate a second time.*

Thus, although the rational-irrational situation is somewhat similar to the left-right situation.

5.3. **The** $\sin(1/x)$ **examples.**

(1) The $\sin(1/x)$ and related examples are somewhat tricky because the function definition differs at an *isolated point*, namely 0.
(2) To calculate any limit or derivative at a point other than 0, we can do formal computations. However, to calculate the derivative at 0, we *must* use the definition of derivative as a limit of a difference quotient.
(3) For all the facts below, the qualitative conclusions hold if we replace sin by cos. The expressions for the derivatives change, but we haven't included those expressions below anyway.
(4) The function $f_0(x) := \{ \begin{array}{rl} \sin(1/x), & x \neq 0 \\ 0, & x = 0 \end{array}$ satisfies the intermediate value property but is not continuous at 0. At all other points, it is infinitely differentiable and we can calculate the derivative formally.
(5) The function $f_1(x) := \{ \begin{array}{rl} x\sin(1/x), & x \neq 0 \\ 0, & x = 0 \end{array}$ is continuous but not differentiable at 0. We can see this from the pinching theorem – it is pinched between $|x|$ and $-|x|$. f_1 is infinitely differentable at all points other than 0.
(6) The function $f_2(x) := \{ \begin{array}{rl} x^2\sin(1/x), & x \neq 0 \\ 0, & x = 0 \end{array}$ is differentiable at 0, and infinitely differentiable everywhere other than 0, but the derivative is not continuous at 0. The limit $\lim_{x \to 0} f_2'(x)$ does not exist. Note that f_2' is defined everywhere and satisfies the intermediate value property but is not continuous.
(7) The function $f_3(x) := \{ \begin{array}{rl} x^3\sin(1/x), & x \neq 0 \\ 0, & x = 0 \end{array}$ is continuously differentiable but not twice differentiable at 0, and infinitely differentiable everywhere other than 0.

5.4. **Testing hypotheses about functions.** Many of you have faced tricky quiz problems where you're asked to determine whether certain general facts about functions are true. Ideally, to show that a general fact is true, you try to give a *generic* proof. However, to show that a general fact is not true, you usually need to come up with a counterexample.

Since the quiz problems are meant to be tricky, I typically choose at least a few options where the most obvious examples you choose don't give counterexamples. However, you can be cleverer still and try to understand how to find good counterexamples. Here are some general tips in that direction:

(1) Start trying to prove (in a rough sense) the statement to be true. Locate the *precise step* where your proof encounters an obstacle. What additional assumption do you need to make here? Try to think of an example of a function that violates this additional assumption.
(2) When the question involves one-sidedness, try both increasing and decreasing functions. Try functions that increase on part of the domain and decrease on another part. Also, try piecewise behavior.

(3) Wherever it makes sense, think of functions defined differently on the rationals and irrationals. This is particularly helpful to get functions that are well behaved at only a handful of points (e.g., continuous at only 14 points, or differentiable at only 11 points).
(4) Wherever it makes sense, think of the $\sin(1/x)$ examples. This is particularly helpful to get functions that are: (i) badly behaved at only one point, and (ii) give examples to show that the implications continuously differentiable $\implies$ differentiable $\implies$ continuous $\implies$ intermediate value property are strict.
(5) If the property that you are interested in (e.g., being periodic, or polynomial) remains preserved on adding a constant function, add a whacko constant function and see if things still hold up.
(6) For functions on intervals extending to infinity in one or both directions, think of examples where the function approaches but does not reach a value. For instance, $1/x^2$ approaches 0 as x approaches infinity, but does not reach it. This is useful for showing that the analogue of the extreme value theorem does not hold for intervals stretching out to infinity in one or both directions.

6. High yield practice

These are areas where practice shortly before the test should offer high yield. These are things that you're probably not yet very good at, but where being good gives you that extra edge:
(1) $\epsilon - \delta$ proofs for quadratic and piecewise linear.
(2) Limit computations for trigonometric functions, particularly those involving chaining. (Refer to the notes on "trigonometric limits and derivatives" for a number of computational techniques).
(3) Converting a definition involving max and min into a piecewise definition.
(4) Piecewise functions (see all the items listed under piecewise functions in "Tricky topics").

7. Quickly

7.1. **Arithmetic.** You should be able to:

- Do quick arithmetic involving fractions.
- Remember $\sqrt{2}$, $\sqrt{3}$, and π to at least two digits.
- Sense when an expression will simplify to 0.
- Compute approximate values for square roots of small numbers, π and its multiples, etc., so that you are able to figure out, for instance, whether $\pi/4$ is smaller or bigger than 1, or two integers such that $\sqrt{39}$ is between them.

7.2. **Computational algebra.** You should be able to:

(1) Add, subtract, and multiply polynomials.
(2) Factorize quadratics or determine that the quadratic cannot be factorized.
(3) Factorize a cubic if you know one of its linear factors (necessary for limit computations).
(4) Do polynomial long division (not usually necessary, but helpful).
(5) Solve simple inequalities involving polynomial and rational functions once you've obtained them in factored form.

7.3. **Computational trigonometry.** You should be able to:

(1) Determine the values of sin and cos at multiples of $\pi/2$.
(2) Determine the intervals where sin and cos are positive and negative.
(3) Recall the values of sin and cos at $\pi/6$, $\pi/4$, and $\pi/3$.

7.4. **Computational limits.** You should be able to: size up a limit, determine whether it is of the form that can be directly evaluated, of the form that we already know does not exist, or indeterminate.

7.5. **Computational differentiation.** You should be able to:

(1) Differentiate a polynomial (written in expanded form) on sight (without rough work).
(2) Differentiate a polynomial (written in expanded form) twice (without rough work).
(3) Differentiate sums of powers of x on sight (without rough work).
(4) Differentiate rational functions with a little thought.
(5) Do multiple differentiations of expressions whose derivative cycle is periodic, e.g., $a \sin x + b \cos x$.

(6) Differentiate simple composites without rough work (e.g., $\sin(x^3)$).

7.6. **Being observant.** You should be able to look at a function and:

(1) Sense if it is odd (even if nobody pointedly asks you whether it is).
(2) Sense if it is even (even if nobody asks you whether it is).
(3) Sense if it is periodic and find the period (even if nobody asks you about the period).

7.7. **Graphing.** You should be able to:

(1) Mentally graph a linear function.
(2) Graph a piecewise linear function with some thought.
(3) Mentally graph a quadratic function (very approximately) – figure out conditions under which it crosses the axis etc.
(4) Mentally graph sin and cos, as well as functions of the $A\sin(mx)$ and $A\cos(mx)$.

7.8. **Fancy pictures.** Keep in mind approximate features of the graphs of:

(1) $\sin(1/x)$, $x\sin(1/x)$, $x^2\sin(1/x)$ and $x^3\sin(1/x)$, particularly the behavior near 0.
(2) The Dirichlet function and its variants – functions defined differently for the rationals and irrationals.