CLASS QUIZ SOLUTIONS: SEPTEMBER 26; TOPIC: FUNCTIONS

VIPUL NAIK

1. Performance review

10 people took this 5-question quiz. The performance was as follows:

(1) (A): Everybody got this correct.
(2) (E): Everybody got this correct.
(3) (B): Everybody got this correct.
(4) (B): Everybody got this correct.
(5) (A): 1 person got this incorrect.

There were thus 9 full scores and 1 score of 4.

2. Solutions

(1) Consider the function $f(x) := |x+1| - |x|$. For which of the following values of x is $f(x)$ equal to 0?
   (A) $-\frac{1}{2}$
   (B) $-\frac{1}{3}$
   (C) 0
   (D) $\frac{1}{3}$
   (E) $\frac{1}{2}$

   Answer: Option (A)

   Explanation: When we set $x = -1/2$, we get $f(x) = |(-1/2) + 1| - |-1/2| = |1/2| - |-1/2|$, which becomes $1/2 - 1/2$, which is equal to 0.

   We can also solve the equation formally, but this is a little trickier, and we will get to it at a later stage.

   The other choices: All the other choices are incorrect:
   Option (B): $f(-1/3) = 2/3 - 1/3 = 1/3$.
   Option (C): $f(0) = 1 - 0 = 1$.
   Option (D): $f(1/3) = 4/3 - 1/3 = 1$.
   Option (E): $f(1/2) = 3/2 - 1/2 = 1$.

   Performance review: Everybody got it correct.

   Historical note: When this same quiz question was asked last year, everybody got it correct.

(2) Consider the function $f(x) := x^2 + 1$. What is the polynomial describing $f(f(x))$?
   (A) $x^2 + 2$
   (B) $x^4 + x^2 + 1$
   (C) $x^4 + x^2 + 2$
   (D) $x^4 + 2x^2 + 1$
   (E) $x^4 + 2x^2 + 2$

   Answer: Option (E)

   Explanation: We have $f(f(x)) = f(x^2+1) = (x^2+1)^2 + 1 = x^4 + 2x^2 + 1 + 1$, which simplifies to option (E).

   The other choices:
   Option (A) is $(x^2 + 1) + 1 = x^2 + 2$. The error here is is not squaring the $x^2 + 1$ expression.
   Option (D) is $(x^2 + 1)^2 = x^4 + 2x^2 + 1$. The error here is in forgetting to add the 1 at the end.
   Options (B) and (C) are like options (D) and (E), with an error in the coefficient of x^2.

   Performance review: Everybody got it correct.

   Historical note: When this same quiz question was asked last year, everybody got it correct.

(3) Consider the function $f(x) := \frac{x}{x^2+1}$. What is $f(f(1))$?
(A) $1/5$
(B) $2/5$
(C) $4/5$
(D) $5/4$
(E) $5/8$

Answer: Option (B)

Explanation: We have:

$$f(1) = \frac{1}{1^2+1} = \frac{1}{2}$$

Thus, $f(f(1)) = f(1/2)$, and we get:

$$f(1/2) = \frac{1/2}{(1/2)^2+1} = \frac{1/2}{5/4} = \frac{1}{2} \cdot \frac{4}{5} = \frac{2}{5}$$

Performance review: Everybody got it correct.

Historical note: When the same question appeared last year, 1 person chose (A), everybody else got this correct.

(4) Consider the function $f(x) := x + 1$. What is $f(f(x))$?
(A) x
(B) $x + 2$
(C) $2x + 1$
(D) $(x + 1)^2$
(E) $x^2 + 1$

Answer: Option (B)

Explanation: We have $f(f(x)) = f(x + 1) = (x + 1) + 1 = x + 2$.

Performance review: Everybody got it correct.

Historical note: When this same quiz question was asked last year, everybody got it correct.

(5) If a circle has radius r, the area of the circle is πr^2. What is the area of a circle with diameter d?
(A) $\pi d^2/4$
(B) $\pi d^2/2$
(C) πd^2
(D) $2\pi d^2$
(E) $4\pi d^2$

Answer: Option (A)

Explanation: The diameter is twice the radius, so the radius is half the diameter, i.e., $r = d/2$. Plugging this in, we get that the area is:

$$\pi r^2 = \pi(d/2)^2 = \pi d^2/4$$

The other choices: Option (B) is the best distractor. It could arise if we forget to square the 2 in the denominator in the above calculation.

The other options could arise through erroneous starting assumptions such as $r = d$ or $r = 2d$.

Performance review: 1 person chose (B), the best distractor. Everybody else got it correct.

Historical note: When this same quiz question was asked last year, everybody got it correct.

CLASS QUIZ SOLUTIONS: SEPTEMBER 28; TOPIC: FUNCTIONS

VIPUL NAIK

1. Performance review

12 students submitted solutions. Here is the score distribution:

(1) Score of 0: Nobody.
(2) Score of 1: Nobody.
(3) Score of 2: 2 persons.
(4) Score of 3: 2 persons.
(5) Score of 4: 5 persons.
(6) Score of 5: 3 persons.

The mean score was 3.75 and the median was 4.

If you scored poorly on this quiz, there is no need to get discouraged, since these question types are probably new for you even if you're familiar with the material. Please take the time to go through the solutions so that you are able to get similar questions correct if you see them in the future.

Note: You are allowed and encouraged to discuss/collaborate with others on some questions (the star-marked ones) but you should ultimately put the solution that *best fits your conscience* and not simply go with the crowd against what you believe to be the correct answer.

Here is the summary of problem wise scores:

(1) Option (E): 9/12 correct.
(2) (*) Option (C): 11/12 correct. *Magic of collaboration?*
(3) Option (D): 12/12 correct.
(4) (*) Option (A): 8/12 correct.
(5) (*) Option (E): 5/12 correct. *Needs review!*

More details in the next section.

2. Solutions

(1) Suppose f and g are functions from $\mathbb{R}$ to $\mathbb{R}$. Suppose both f and g are even, i.e., $f(x) = f(-x)$ for all $x \in \mathbb{R}$ and $g(x) = g(-x)$ for all $x \in \mathbb{R}$. Which of the following is *not* guaranteed to be an even function from the given information?

Note: For this question, it is possible to solve the question by taking a few simple examples. You're free to do this, but the recommended method for tackling the question is to handle it **abstractly**, *i.e., try to prove or disprove in general for each function whether it is even.*

(A) $f + g$, i.e., the function $x \mapsto f(x) + g(x)$
(B) $f - g$, i.e., the function $x \mapsto f(x) - g(x)$
(C) $f \cdot g$, i.e., the function $x \mapsto f(x)g(x)$
(D) $f \circ g$, i.e., the function $x \mapsto f(g(x))$
(E) None of the above, i.e., they are all guaranteed to be even functions.

Answer: Option (E)

Explanation: We show that $f+g$, $f-g$, $f \cdot g$, and $f \circ g$ are all even. Below are given short versions of "proofs" of these facts.

$f + g$ is even because

$$(f + g)(-x) = f(-x) + g(-x) = f(x) + g(x) = (f + g)(x)$$

Hence $(f + g)(-x) = (f + g)(x)$ for all x.

$f - g$ is even because

$$(f - g)(-x) = f(-x) - g(-x) = f(x) - g(x) = (f - g)(x)$$

$f \cdot g$ is even because

$$(f \cdot g)(-x) = f(-x)g(-x) = f(x)g(x) = (f \cdot g)(x)$$

$f \circ g$ is even because:

$$(f \circ g)(-x) = f(g(-x)) = f(g(x)) = (f \circ g)(x)$$

Note that $f \circ g$ is somewhat special, because for this case, we only use that g is even – we do not use or require f to be even.

Performance review: 9 out of 12 people got this correct. 3 chose (D).

Historical note (last year): 11 out of 15 people got this correct. 3 people chose (D) and 1 person chose (B).

(2) (*) Suppose f and g are functions from $\mathbb{R}$ to $\mathbb{R}$. Suppose both f and g are odd, i.e., $f(-x) = -f(x)$ for all $x \in \mathbb{R}$ and $g(-x) = -g(x)$ for all $x \in \mathbb{R}$. Which of the following is *not* guaranteed to be an odd function from the given information?

Note: For this question, it is possible to solve the question by taking a few simple examples. You're free to do this, but the recommended method for tackling the question is to handle it **abstractly***, i.e., try to prove or disprove in general for each function whether it is odd.*

(A) $f + g$, i.e., the function $x \mapsto f(x) + g(x)$
(B) $f - g$, i.e., the function $x \mapsto f(x) - g(x)$
(C) $f \cdot g$, i.e., the function $x \mapsto f(x)g(x)$
(D) $f \circ g$, i.e., the function $x \mapsto f(g(x))$
(E) None of the above, i.e., they are all guaranteed to be odd functions.

Answer: Option (C)

Explanation: An example is when $f(x) := x$ and $g(x) := x$. Both f and g are odd functions. But the product $f \cdot g$ the function $x \mapsto x^2$, which is an even function.

The other choices:

Option (A): If f and g are both odd, then $f + g$ has to be odd. Here's why:

$$(f + g)(-x) = f(-x) + g(-x) = -f(x) + (-g(x)) = -[f(x) + g(x)] = -(f + g)(x)$$

Option (B): If f and g are both odd, then $f - g$ has to be odd. Here's why:

$$(f - g)(-x) = f(-x) - g(-x) = -f(x) - (-g(x)) = -[f(x) - g(x)] = -(f - g)(x)$$

Option (D): If f and g are both odd, then $f \circ g$ has to be odd. Here's why:

$$(f \circ g)(-x) = f(g(-x)) = f(-g(x)) = -f(g(x)) = -(f \circ g)(x)$$

Option (E) is ruled out because option (C) works.

Performance review: 11 out of 12 got this correct. 1 chose (D). *Performance this year was much better than last year, possibly due to the benefits of collaboration?*

Historical note (last year): 7 out of 15 people got this correct. 5 people chose (E) and 3 people chose (D).

(3) For which of the following pairs of polynomial functions f and g is it true that $f \circ g \neq g \circ f$?

(A) $f(x) := x^2$ and $g(x) := x^3$
(B) $f(x) := x + 1$ and $g(x) := x + 2$
(C) $f(x) := x^2 + 1$ and $g(x) := x^2 + 1$
(D) $f(x) := -x$ and $g(x) := x^2$
(E) $f(x) := -x$ and $g(x) := x^3$

Answer: Option (D)

Explanation: For option (D), $f(g(x)) = -x^2$ and $g(f(x)) = (-x)^2 = x^2$. The two polynomials take distinct values for all nonzero x, hence they are not equal as functions.

The other choices:

Option (A): $f(g(x)) = (x^3)^2 = x^{3\cdot 2} = x^6$. Similarly $g(f(x)) = (x^2)^3 = x^{2\cdot 3} = x^6$. So in this case $f \circ g = g \circ f$.

Option (B): $f(g(x)) = (x + 2) + 1 = x + 3$ and $g(f(x)) = (x + 1) + 2 = x + 3$. So in this case $f \circ g = g \circ g$.

Option (C): Here, $f = g$ so both $f \circ g$ and $g \circ f$ are equal to $f \circ f$. Note that we do not need to explicitly compute $f \circ g$ in this case.

Option (E): Here, $f(g(x)) = -x^3$ and $g(f(x)) = (-x)^3 = (-1)^3x^3 = -x^3$.

Additional remark: If $f \circ g = g \circ f$, we say that the functions f and g *commute*. You may have heard about the commutativity law for addition and multiplication. In the case of function composition, commutativity is *not* a law. It holds for some pairs of functions (such as options (A), (B), (D), (E) here) and not for others (such as option (C)).

Note that for the function $f(x) := -x$, a function h commutes with f if and only if h is an odd function. Thus, in option (E), the cube map is an odd function. And option (D) fails because the square map is *not* an odd function.

Performance review: Everybody(12 out of 12) got this correct.

Historical note (last year): 14 out of 15 people got this correct. 1 person chose (B).

Action point: Even if you got this correct, it may be helpful to try to understand how to more quickly see that for all the other pairs, $f \circ g = g \circ f$.

(4) (*) Which of the following functions is *not* periodic?
(A) $\sin(x^2)$
(B) $\sin^2 x$
(C) $\sin(\sin x)$
(D) $\sin(x + 13)$
(E) $(\sin x) + 13$

Answer: Option (A)

Explanation: It's somewhat hard to show that $\sin(x^2)$ is not a periodic function. (This will be an advanced problem in a subsequent homework).

On the other hand, it is easy to solve this problem by elimination, since the other four options are periodic, as explained below.

The other choices:

Option (B): $\sin^2$ is periodic and has period π. This follows from the fact that $\sin(\pi + x) = -\sin x$. Even if you don't notice that the period is π, you can still deduce 2π-periodicity from the fact that sin is 2π-periodic.

Option (C): $\sin \circ \sin$ is 2π-periodic.

Note that options (B) and (C) both from a more general fact: if $f, g : \mathbb{R} \to \mathbb{R}$ are functions, and g is periodic, so is $f \circ g$, and any choice of $h > 0$ that works for g also works for $f \circ g$ (though the period for $f \circ g$ may be smaller than that for g)

Option (D): This is 2π-periodic. In graphical terms, it is obtained by shifting the graph of the sin function 13 units to the left, and the periodicity pattern is unaffected.

Option (E): This is 2π-periodic. Adding a constant function (in this case, 13) to a periodic function (in this case, sin) gives a periodic function with the same period.

Performance review: 8 out of 12 got this correct. 2 each chose (B) and (C).

Historical note (last year): 8 out of 15 people got this correct. 3 people chose (B), 3 people chose (C), and 1 person chose (D).

Action point: Please make sure you understand the reasoning for why the other functions are periodic. You may also experiment with plotting the graphs of these functions using Mathematica or a graphing calculator.

(5) (*) What is the domain of the function $\sqrt{1 - x} + \sqrt{x - 2}$? Here, domain refers to the *largest set* on which the function can be defined.

(A) $(1, 2)$
(B) $[1, 2]$
(C) $(-\infty, 1) \cup (2, \infty)$
(D) $(-\infty, 1] \cup [2, \infty)$
(E) None of the above

Hint: Think clearly, first about what the domain of each of the two functions being added is, and then about whether you need to take the union or the intersection of the domains of the individual functions.

Answer: Option (E)

Explanation: For $\sqrt{1-x}$ to be defined, we need $1 - x \geq 0$, so $x \leq 1$. For $\sqrt{x-2}$ to be defined, we need $2 - x \geq 0$, so $x \geq 2$. Thus, we require that $x \leq 1$ and $x \geq 2$ hold *simultaneously*. In set theory terms, we need to take the *intersection* of the solution sets to $x \leq 1$ (which is $(-\infty, 1]$) and to $x \geq 2$ (which is $[2, \infty)$).

The two conditions cannot hold together, i.e., the intersection of the solutions for the two constraints is empty. Hence, the domain of the function is in fact empty, i.e., the function is defined *nowhere*.

The other choices: Options (C) and (D) are the most sophisticated distractors. Option (D) is the union of the domains of $\sqrt{1-x}$ and $\sqrt{x-2}$. However, what we need here is the *intersection* of the domains, not the union.

Performance review: 5 out of 12 got this correct. 7 chose (D), which is the *union* rather than the *intersection*. So, people fell for the sophisticated distractors, not the silly ones.

Historical note (last year): 8 out of 15 people got this correct. 5 people chose (D), 1 person chose (B), and 1 person chose (C).

Action point: Please make sure you understand why we need to *intersect* the domains of f and g rather than take the union. This goes back to the fact discussed in class that the domain of $f + g$ is the intersection of the domains of f and g.

CLASS QUIZ SOLUTIONS: SEPTEMBER 30: LIMITS

VIPUL NAIK

1. Performance review

12 people took this quiz. The score distribution was as follows:

- Score of 2: 8 people.
- Score of 3: 4 people.

The median score was 2 and the mean score was 2.33.
Here is the information by question:

(1) Option (A): 11 people got this correct.
(2) Option (D): 8 people got this correct. *Although many of you got this correct, it's likely that some of you did so through educated guesswork, so I recommend you go through the solution, which is not completely straightforward.*
(3) Option (B): 9 people got this correct.

More details in the next section.

2. Solutions

(1) (**) We call a function f left continuous on an open interval I if, for all $a \in I$, $\lim_{x \to a^-} f(x) = f(a)$. Which of the following is an example of a function that is left continuous but not continuous on $(0, 1)$?

(A) $f(x) := \{ \begin{array}{rl} x, & 0 < x \le 1/2 \\ 2x, & 1/2 < x < 1 \end{array}$

(B) $f(x) := \{ \begin{array}{rl} x, & 0 < x < 1/2 \\ 2x, & 1/2 \le x < 1 \end{array}$

(C) $f(x) := \{ \begin{array}{rl} x, & 0 < x \le 1/2 \\ 2x - (1/2), & 1/2 < x < 1 \end{array}$

(D) $f(x) := \{ \begin{array}{rl} x, & 0 < x < 1/2 \\ 2x - (1/2), & 1/2 \le x < 1 \end{array}$

(E) All of the above

Answer: Option (A)

Explanation: Note that in all four cases, the two pieces of the function are continuous. Thus, the relevant questions are: (i) do the two definitions agree at the point where the definition changes (in all four cases here, 1/2)? and (ii) is the point (in all cases, 1/2) where the definition changes included in the left or the right piece?

For options (C) and (D), the definitions on the left and right piece agree at 1/2. Namely the function x and $2x - (1/2)$ both take the value 1/2 at the domain point 1/2. Thus, options (C) and (D) both define continuous functions (in fact, the same continuous function).

This leaves options (A) and (B). For these, the left definition x and the right definition $2x$ do not match at 1/2: the former gives 1/2 and the latter gives 1. In other words, the function has a jump discontinuity at 1/2. Thus, (ii) becomes relevant: is 1/2 included in the left or the right definition?

For option (A), 1/2 is included in the left definition, so $f(1/2) = 1/2 = \lim_{x \to 1/2^-} f(x)$. On the other hand, $\lim_{x \to 1/2^+} f(x) = 1$. Thus, the f in option (A) is left continuous but not right continuous.

For option (B), 1/2 is included in the right definition, so $f(1/2) = 1$ and f is right continuous but not left continuous at 1/2.

Performance review: 11 out of 12 people got this correct. 1 chose (C).

Historical note (last year): 6 out of 13 people got this correct. 6 people chose option (E) – I'm not sure why this option was so popular. 1 person chose option (C) but was quite close to choosing (A).

Action point: Please read through the lecture notes on Chalk (title "informal introduction to limits" – these roughly correspond to today's lecture), as well as any notes you took during class discussion today, very carefully, till you are completely confusion-free on the issues of left and right limits and continuity. This is the kind of question that, once you are thorough with the definitions, you should be able to get correctly. In other words, while the question is "hard" today, I expect it to be in the "moderately easy" category by the time you take the first midterm.

(2) (**) Suppose f and g are functions $(0,1)$ to $(0,1)$ that are both left continuous on $(0,1)$. Which of the following is *not* guaranteed to be left continuous on $(0,1)$?
(A) $f+g$, i.e., the function $x \mapsto f(x)+g(x)$
(B) $f-g$, i.e., the function $x \mapsto f(x)-g(x)$
(C) $f \cdot g$, i.e., the function $x \mapsto f(x)g(x)$
(D) $f \circ g$, i.e., the function $x \mapsto f(g(x))$
(E) None of the above, i.e., they are all guaranteed to be left continuous functions

Answer: Option (D)

Explanation: We need to construct an explicit example, but we first need to do some theoretical thinking to motivate the right example. The full reasoning is given below.

Motivation for example: Left hand limits split under addition, subtraction and multiplication, so options (A)-(C) are guaranteed to be left continuous, and are thus false. This leaves the option $f \circ g$ for consideration. Let us look at this in more detail.

For $c \in (0,1)$, we want to know whether:

$$\lim_{x \to c^-} f(g(x)) \stackrel{?}{=} f(g(c))$$

We do know, by assumption, that, as x approaches c from the left, $g(x)$ approaches $g(c)$. However, we do not know whether $g(x)$ approaches $g(c)$ from the left or the right or in oscillatory fashion. If we could somehow guarantee that $g(x)$ approaches $g(c)$ from the left, then we would obtain that the above limit holds. However, the given data does not guarantee this, so (D) is false.

We need to construct an example where g is *not* an increasing function. In fact, we will try to pick g as a decreasing function, so that when x approaches c from the left, $g(x)$ approaches $g(c)$ from the right. As a result, when we compose with f, the roles of left and right get switched. Further, we need to construct f so that it is left continuous but not right continuous.

Explanation with example: Consider the case where, say:

$$f(x) := \{ \begin{array}{ll} 1/3, & 0 < x \le 1/2 \\ 2/3, & 1/2 < x < 1 \end{array}$$

and

$$g(x) := 1 - x$$

Note that both functions have range a subset of $(0,1)$.

Composing, we obtain that:

$$f(g(x)) = \{ \begin{array}{ll} 2/3, & 0 < x < 1/2 \\ 1/3, & 1/2 \le x < 1 \end{array}$$

f is left continuous but not right continuous at $1/2$, whereas $f \circ g$ is right continuous but not left continuous at $1/2$.

Performance review: 8 out of 12 got this correct. 4 chose (E).

Historical note (last year): 4 out of 13 people got this correct. 9 people chose option (E). This is understandable, because if you look only at the obvious examples (all of which are increasing functions), you are likely to think that $f \circ g$ must be left continuous. If you got this question right for the right reasons, congratulate yourself.

Action point: We will emphasize the moral of this problem in a class in the near future. When we discuss the theorems involving limits, we will note that the theorems on sums, differences, products, etc. also hold for one-sided limits (i.e., each of the theorems holds for left hand limits and each of the theorems holds for right hand limits). However, the theorem on compositions for limits does not hold for one-sided limits, unless we make additional assumptions. I hope you will never forget this point (or at least, not till the end of the winter quarter).

(3) (*) Consider the function:

$$f(x) := \{ \begin{array}{rl} x, & x \text{ rational} \\ 1/x, & x \text{ irrational} \end{array}$$

What is the set of all points at which f is continuous?

(A) $\{0, 1\}$
(B) $\{-1, 1\}$
(C) $\{-1, 0\}$
(D) $\{-1, 0, 1\}$
(E) f is continuous everywhere

Answer: Option (B)

Explanation: In this interesting example, instead of a *left* versus *right* split, we are splitting the domain into rationals and irrationals. For the overall limit to exist at c, we need that: (i) the limit for the function as defined for rationals exists at c, (ii) the limit for the function as defined for irrationals exists at c, and (iii) the two limits are equal.

Note that regardless of whether the point c is rational or irrational, we need *both* the rational domain limit and the irrational domain limit to exist and be equal at c. This is because rational numbers are surrounded by irrational numbers and vice versa – both rational numbers and irrational numbers are dense in the reals – hence at any point, we care about the limits restricted to the rationals as well as the irrationals.

The limit for rationals exists for all c and equals the value c. The limit for irrationals exists for all $c \neq 0$ and equals the value $1/c$. For these two numbers to be equal, we need $c = 1/c$. Solving, we get $c^2 = 1$ so $c = \pm 1$.

Performance review: 9 out of 12 got this correct. 3 chose (E).

Historical note (last year): 5 out of 13 people got this correct. 3 people chose (D), 3 people chose (A), and 1 person each chose (C) and (E). Some of the people who chose (D) wrote "all rationals", so they probably thought that the correct answer is "all rationals" but it was not one of the options.

Action point: Getting this correct requires a thorougher definition of limit than the purely intuitive one. Like the $\sin(1/x)$-based functions, these functions are fascinating precisely because of the lack of clarity in what it means for such a function to have a limit. In a couple of weeks, after you have dealt more with functions defined differently for the rationals and irrationals, *and* seen the $\epsilon - \delta$ definition of limit, you will be in a much better position to tackle this question. By the time of the first midterm, this question should be in the "moderately easy" category.

CLASS QUIZ SOLUTIONS: OCTOBER 3: LIMITS

MATH 152, SECTION 55 (VIPUL NAIK)

1. Performance review

12 people took this quiz. The score distribution was as follows:

- Score of 0: 1 person
- Score of 1: 2 people
- Score of 2: 5 people
- Score of 3: 3 people
- Score of 4: 1 person

Here are the answers:

(1) Option (C): 8 people
(2) Option (C): 2 people. *Please review!*
(3) Option (B): 8 people
(4) Option (C): 7 people

2. Solutions

(1) Which of these is the correct interpretation of $\lim_{x\to c} f(x) = L$ in terms of the definition of limit?
  (a) For every $\alpha > 0$, there exists $\beta > 0$ such that if $0 < |x - c| < \alpha$, then $|f(x) - L| < \beta$.
  (b) There exists $\alpha > 0$ such that for every $\beta > 0$, and $0 < |x - c| < \alpha$, we have $|f(x) - L| < \beta$.
  (c) For every $\alpha > 0$, there exists $\beta > 0$ such that if $0 < |x - c| < \beta$, then $|f(x) - L| < \alpha$.
  (d) There exists $\alpha > 0$ such that for every $\beta > 0$ and $0 < |x - c| < \beta$, we have $|f(x) - L| < \alpha$.

  Answer: Option (C)

  Explanation: α plays the role of ϵ and β plays the role of δ.

  Performance review: 8 out of 12 got this correct. 2 chose (B), 1 each chose (A) and (E).

  Historical note (last year): 9 out of 12 people got this correct. 2 people chose (B) and 1 person chose (D).

  Action point: If you got this correct, that means that you are not completely fixated on the letters ϵ and δ. This is good news, because it is important to concentrate on the substantive meaning rather than get caught up with a name. If you had difficulty with this, make sure you can understand it now.

(2) Suppose $f : \mathbb{R} \to \mathbb{R}$ is a function. Which of the following says that f does not have a limit at any point in $\mathbb{R}$ (i.e., there is no point $c \in \mathbb{R}$ for which $\lim_{x\to c} f(x)$ exists)?
  (A) For every $c \in \mathbb{R}$, there exists $L \in \mathbb{R}$ such that for every $\epsilon > 0$, there exists $\delta > 0$ such that for all x satisfying $0 < |x - c| < \delta$, we have $|f(x) - L| \geq \epsilon$.
  (B) There exists $c \in \mathbb{R}$ such that for every $L \in \mathbb{R}$, there exists $\epsilon > 0$ such that for every $\delta > 0$, there exists x satisfying $0 < |x - c| < \delta$ and $|f(x) - L| \geq \epsilon$.
  (C) For every $c \in \mathbb{R}$ and every $L \in \mathbb{R}$, there exists $\epsilon > 0$ such that for every $\delta > 0$, there exists x satisfying $0 < |x - c| < \delta$ and $|f(x) - L| \geq \epsilon$.
  (D) There exists $c \in \mathbb{R}$ and $L \in \mathbb{R}$ such that for every $\epsilon > 0$, there exists $\delta > 0$ such that for all x satisfying $0 < |x - c| < \delta$, we have $|f(x) - L| \geq \epsilon$.
  (E) All of the above.

  Answer: Option (C)

  Explanation: Our statement should be that *every* c has no limit. In other words, for *every* c and *every* L, it is *not* true that $\lim_{x\to c} f(x) = L$. That's exactly what option (C) says.

  Performance review: 2 out of 12 got this correct. 7 chose (B), 2 chose (D), 1 chose (A).

Historical note (last year): 10 out of 12 people got this correct. 1 person each chose (B) and (E).

Note on performance discrepancy with last year: I now remember that last year I had discussed the idea behind this very question in the same class as the quiz was administered, so students last year had an advantage in doing the quiz.

Action point: If you got this correct, great! Do make sure you understand this thoroughly – review the $\epsilon - \delta$ definition yet another time and try to understand how this follows from that definition.

(3) In the usual $\epsilon - \delta$ definition of limit for a given limit $\lim_{x \to c} f(x) = L$, if a given value $\delta > 0$ works for a given value $\epsilon > 0$, then which of the following is true?
  (A) Every smaller positive value of δ works for the same ϵ. Also, the given value of δ works for every smaller positive value of ϵ.
  (B) Every smaller positive value of δ works for the same ϵ. Also, the given value of δ works for every larger value of ϵ.
  (C) Every larger value of δ works for the same ϵ. Also, the given value of δ works for every smaller positive value of ϵ.
  (D) Every larger value of δ works for the same ϵ. Also, the given value of δ works for every larger value of ϵ.
  (E) None of the above statements need always be true.

Answer: Option (B)

Explanation: This can be understood in multiple ways. One is in terms of the prover-skeptic game. A particular choice of δ that works for a specific ϵ also works for larger ϵs, because the function is already "trapped" in a smaller region. Further, smaller choices of δ also work because the skeptic has fewer values of x.

Rigorous proofs are being skipped here, but you can review the formal definition of limit notes if this stuff confuses you.

Performance review: 8 out of 12 got this correct. 3 chose (A), 1 chose (E).

Historical note (last year): 17 out of 26 people got this correct. 5 people chose (A), 3 chose (C), and 1 chose (D).

(4) Which of the following is a correct formulation of the statement $\lim_{x \to c} f(x) = L$, in a manner that avoids the use of ϵs and δs? *Not appeared in previous years*
  (A) For every open interval centered at c, there is an open interval centered at L such that the image under f of the open interval centered at c (excluding the point c itself) is contained in the open interval centered at L.
  (B) For every open interval centered at c, there is an open interval centered at L such that the image under f of the open interval centered at c (excluding the point c itself) contains the open interval centered at L.
  (C) For every open interval centered at L, there is an open interval centered at c such that the image under f of the open interval centered at c (excluding the point c itself) is contained in the open interval centered at L.
  (D) For every open interval centered at L, there is an open interval centered at c such that the image under f of the open interval centered at c (excluding the point c itself) contains the open interval centered at L.
  (E) None of the above.

Answer: Option (C)

Explanation: The "open interval centered at L" describes the "$\epsilon > 0$" part of the definition (where the open interval is the interval $(L-\epsilon, L+\epsilon)$). The "open interval centered at c" describes the "$\delta > 0$" part of the definition (where the open interval is the interval $(c - \delta, c + \delta)$). x being in the open interval centered at c (except the case $x = c$) is equivalent to $0 < |x - c| < \delta$, and $f(x)$ being in the open interval centered at L is equivalent to $|f(x) - L| < \epsilon$.

Performance review: 7 out of 12 got this correct. 2 chose (A), 1 each chose (B), (D), and (E).

Action point: You should master this way of thinking. This is actually the "correct" way of thinking of the definition. You'll see what I mean in future math classes (153 and beyond).

CLASS QUIZ SOLUTIONS: OCTOBER 7: LIMIT THEOREMS

MATH 152, SECTION 55 (VIPUL NAIK)

1. Performance review

12 students took this quiz. The score distribution was as follows:

- Score of 2: 3 people
- Score of 3: 5 people
- Score of 4: 4 people

The mean score was 3.08. Here are the problem-wise answers and scores:

(1) Option (A): 6 people
(2) Option (C): 4 people
(3) Option (D): 10 people
(4) Option (D): 9 people
(5) Option (B): 8 people

More details below.

2. Solutions

(1) (**) Which of the following statements is **always true**?
  (A) The range of a continuous nonconstant function on a closed bounded interval (i.e., an interval of the form $[a, b]$) is a closed bounded interval (i.e., an interval of the form $[m, M]$).
  (B) The range of a continuous nonconstant function on an open bounded interval (i.e., an interval of the form (a, b)) is an open bounded interval (i.e., an interval of the form (m, M)).
  (C) The range of a continuous nonconstant function on a closed interval that may be bounded or unbounded (i.e., an interval of the form $[a, b]$, $[a, \infty)$, $(-\infty, a]$, or $(-\infty, \infty)$) is also a closed interval that may be bounded or unbounded.
  (D) The range of a continuous nonconstant function on an open interval that may be bounded or unbounded (i.e., an interval of the form (a, b),(a, ∞), $(-\infty, a)$, or $(-\infty, \infty)$), is also an open interval that may be bounded or unbounded.
  (E) None of the above.

Answer: Option (A)

Explanation: This is a combination of the extreme-value theorem and the intermediate-value theorem. By the extreme-value theorem, the continuous function attains a minimum value m and a maximum value M. By the intermediate-value theorem, it attains every value between m and M. Further, it can attain no other values because m is after all the minimum and M the maximum.

The other choices:

Option (B): Think of a function that increases first and then decreases. For instance, the function $f(x) := \sqrt{1 - x^2}$ on $(-1, 1)$ has range $(0, 1]$, which is not open. Or, the function $\sin x$ on the interval $(0, 2\pi)$ has range $[-1, 1]$.

Option (C): We can get counterexamples for unbounded intervals. For instance, consider the function $f(x) := 1/x$ on $[1, \infty)$. The range of this function is $(0, 1]$, which is not closed. The idea is that we make the function approach but not reach a finite value as $x \to \infty$ (we'll talk more about this when we deal with asymptotes).

Option (D): The same counterexample as for option (B) works.

Performance review: 6 out of 12 got this correct. 3 chose (C), 2 chose (D), 1 chose (E).

Historical note (last year): 2 out of 11 people got this correct. (C) was the most frequently chosen incorrect answer.

Action point: Please review the statement of the extreme value theorem, as well as understand why all the other examples are incorrect.

(2) (**) Suppose $g : \mathbb{R} \to \mathbb{R}$ is a continuous function such that $\lim_{x\to 0} g(x)/x = A$ for some constant $A \neq 0$. What is $\lim_{x\to 0} g(g(x))/x$?
(A) 0
(B) A
(C) A^2
(D) $g(A)$
(E) $g(A)/A$

Answer: Option (C)

Explanation: We have $\lim_{x\to 0} g(x) = \lim_{x\to 0}(g(x)/x) \lim_{x\to 0} x = A \cdot 0 = 0$.

Also, we have:

$$\lim_{x\to 0} \frac{g(g(x))}{x} = \lim_{x\to 0} \frac{g(g(x))}{g(x)} \lim_{x\to 0} \frac{g(x)}{x}$$

The second limit is A. For the first limit, note that as $x \to 0$, we also have $g(x) \to 0$, so the first limit can be rewritten as $\lim_{y\to 0} g(y)/y$, which is also equal to A. Hence, the overall limit is the product A^2.

Performance review: 4 out of 12 go this correct. 3 each chose (A) and (E), 2 chose (D).

Historical note (last year): 1 out of 12 people got this correct. 5 people chose (D), 2 people each chose (B) and (E), 1 person chose (A), and 1 person left the question blank.

(3) Suppose $I = (a, b)$ is an open interval. A function $f : I \to \mathbb{R}$ is termed *piecewise continuous* if there eixst points $a_0 < a_1 < a_2 < \cdots < a_n$ (dependent on f) with $a = a_0$ and $a_n = b$, such that f is continuous on each interval (a_i, a_{i+1}). In other words, f is continuous at every point in (a, b) except possibly the a_is.

Suppose f and g are piecewise continuous functions on the same interval I (with possibly different sets of a_is). Which of the following is/are guaranteed to be piecewise continuous on I?
(A) $f + g$, i.e., the function $x \mapsto f(x) + g(x)$
(B) $f - g$, i.e., the function $x \mapsto f(x) - g(x)$
(C) $f \cdot g$, i.e., the function $x \mapsto f(x)g(x)$
(D) All of the above
(E) None of the above

Answer: Option (D)

Explanation: We take the points where f is possibly discontinuous and the points where g is possible discontinuous, and we take the union of these sets of points. We get a new finite set of points. Note that everywhere except these points, both f and g are continuous, hence $f + g$, $f - g$, and $f \cdot g$ are all continuous.

A numerical illustration might help here. (Note, however, that there is nothing special about the numbers). Suppose $a = 1$ and $b = 2$. Let's say that f is continuous on $(1, 1.5)$ and $(1.5, 2)$, so it is possibly discontinuous at 1.5. Suppose g is continuous on $(1, \sqrt{2})$, $(\sqrt{2}, \sqrt{3})$ and $(\sqrt{3}, 2)$, so the points where it may be discontinuous are $\sqrt{2}$ and $\sqrt{3}$.

We now take the union of the points of discontinuity of f and g. We get the points 1.5, $\sqrt{2}$, and $\sqrt{3}$. Recall that $\sqrt{2} \approx 1.414 < 1.5$ while $\sqrt{3} \approx 1.732 > 1.5$, so rearranging in increasing order, we get $1 < \sqrt{2} < 1.5 < \sqrt{3} < 2$. We can now see that $f + g$, $f - g$ and $f \cdot g$ are all continuous on the intervals $(1, \sqrt{2})$, $(\sqrt{2}, 1.5)$, $(1.5, \sqrt{3})$ and $(\sqrt{3}, 2)$.

Memory lane: This is the idea of *breaking up the domains for two piecewise definned functions in the same manner* so as to be able to add, subtract, and multiply them. You have seen a problem with this theme in Homework 1, Problem 8 (Exercise 1.7.14 of the book, Page 46). There, goal was to compute $f + g$, $f - g$, and $f \cdot g$ with f and g given piecewise with different domain breakdowns. Here, our goal is to pontificate about continuity, but the idea is the same.

Future teaser: This idea of partitioning an interval into sub-intervals by choosing some points keeps coming up. Further, the idea of combining two partitions of the same interval into a finer partition that refines both of them will also come up. Specifically, both these ideas turn up when we try to define the integral of a continuous (or piecewise continuous) function on an interval.

Performance review: 10 out of 12 got this correct. 2 chose (E).

Historical note (last year): 9 out of 11 people got this correct. 1 person chose (C) and 1 person chose (E).

Action point: Whether or not you got this correct, make sure that you *now* understand the logic behind it. This idea is extremely important in the future.

(4) Suppose f and g are everywhere defined and $\lim_{x\to 0} f(x) = 0$. Which of these pieces of information is **not sufficient** to conclude that $\lim_{x\to 0} f(x)g(x) = 0$?
(A) $\lim_{x\to 0} g(x) = 0$.
(B) $\lim_{x\to 0} g(x)$ is a constant not equal to zero.
(C) There exists $\delta > 0$ and $B > 0$ such that for $0 < |x| < \delta$, $|g(x)| < B$.
(D) $\lim_{x\to 0} g(x) = \infty$, i.e., for every $N > 0$ there exists $\delta > 0$ such that if $0 < |x| < \delta$, then $g(x) > N$.
(E) None of the above, i.e., they are all sufficient to conclude that $\lim_{x\to 0} f(x)g(x) = 0$.

Answer: Option (D)

Explanation: If $f(x) \to 0$ and $g(x) \to \infty$, then the limit of $f(x)g(x)$ is indeterminate. It may be 0, finite, infinite, or oscillatory. For instance, if $f(x) = x^2$ and $g(x) = 1/x^2$, then the limit of $f(x)g(x)$ is 1. Thus, we cannot conclude that the limit of the product is 0.

Memory lane: Routine Problem 5 on Homework 2 (Exercise 2.3.3 of the book, Page 79) explores a similar theme. The new ingredient here is that, in cases where $f(x)$ goes to zero and $g(x)$ does not have a limit but is still bounded, we *can* say that the product goes to zero.

The other choices:

Option (A) is sufficient because the limit of the sums is the sum of the limits.

Option (B) is sufficient for the same reason.

Option (C) is a little trickier to justify. Here, what we are saying is that $\lim_{x\to 0} f(x) = 0$ and, for x close enough to 0, g is bounded, though it need not have a limit. The bound here is B. In particular, what this is saying is that if $0 < |x| < \delta$, then $g(x)$ is between $-B$ and B.

Thus, we can see that:

$$-Bf(x) \leq f(x)g(x) \leq Bf(x) \; \forall \; 0 < |x| < \delta$$

We now note that both $-Bf(x)$ and $Bf(x)$ tend to 0 as $x \to 0$. Hence, by the pinching theorem, $f(x)g(x) \to 0$.

Examples for g: One example of such a function g is the Dirichlet function, i.e., $g(x)$ is 1 if x is rational and 0 if x is irrational. Clearly, the Dirichlet function is bounded near 0 (in fact, it is universally bounded). If we set $f(x) := x$, then $f(x)g(x) = \{ \begin{matrix} x, & x \text{ rational} \\ 0, & x \text{ irrational} \end{matrix}$, and the limit of this function at 0 is 0. Incidentally, Advanced Problem 3 of Homework 2 (Exercise 2.2.54 of the book, Page 72) asked you to give an explicit $\epsilon - \delta$ proof of this fact.

Another example for g is the $\sin(1/x)$ function. This function oscillates between -1 and 1, hence does not converge to a limit as $x \to 0$. However, it is bounded. Thus, if we have $f(x) := x$, the function $x \sin(1/x)$ must converge to 0 as x goes to 0. This function appears in Advanced Problem 4 of Homework 3 (Exercise 3.6.67 of the book, Page 146).

Performance review: 10 out of 12 got this correct. 1 each chose (A), (C), and (E).

Historical note (last year): 8 out of 11 people got this correct. 1 person each chose (A), (B), and (C).

Action point: You should understand this, but don't have to worry too much about it for now. We will cover these issues in more detail later.

(5) f and g are functions defined for all real values. c is a real number. Which of these statements is ***not* necessarily true**?
(A) If $\lim_{x\to c^-} f(x) = L$ and $\lim_{x\to c^-} g(x) = M$, then $\lim_{x\to c^-}(f(x) + g(x))$ exists and is equal to $L + M$.
(B) If $\lim_{x\to c^-} g(x) = L$ and $\lim_{x\to L^-} f(x) = M$, then $\lim_{x\to c^-} f(g(x)) = M$.

(C) If there exists an open interval containing c on which f is continuous and there exists an open interval containing c on which g is continuous, then there exists an open interval containing c on which $f + g$ is continuous.

(D) If there exists an open interval containing c on which f is continuous and there exists an open interval containing c on which g is continuous, then there exists an open interval containing c on which the product $f \cdot g$ (i.e., the function $x \mapsto f(x)g(x)$) is continuous.

(E) None of the above, i.e., they are all necessarily true.

Answer: Option (B)

Explanation: This is the cliched fact that composition results do not hold for one-sided limits. The main reason is that when we compose, we need the inner function of the composition to approach the limit *from the correct side* in order for the result to go through. Thus, in this case, for instance, the result would be true if the function g were strictly increasing on the immediate left of c.

Memory lane: We already saw this fact in the October 1 quiz on limits, Problem 2. Please review the solution to that (where we've also given an explicit example).

The other choices:

Option (A) is the sum theorem for one-sided limits: the limit of the sum is the sum of the limits.

For options (C) and (D), note that if f is continuous on one open interval containing c and g is continuous on another open interval containing c, then *both* f and g are continuous on the *intersection* of the two open intervals containing c (which is also an open interval containing c). Thus, $f+g$ is also continuous on this intersection. This is analogous to the trick we often use of picking $\delta = \min\{\delta_1, \delta_2\}$ in $\epsilon - \delta$ proofs for piecewise functions.

Performance review: 8 out of 12 got this correct. 2 chose (C) and 2 chose (E).

Historical note (last year): 9 out of 11 people got this correct. 1 person each chose (A) and (E).

CLASS QUIZ SOLUTIONS: OCTOBER 10: DERIVATIVES

MATH 152, SECTION 55 (VIPUL NAIK)

1. Performance review

12 people took the quiz. The score distribution was as follows:

- Score of 2: 2 people
- Score of 3: 4 people
- Score of 4: 6 people

The mean score was 3.33. Here are the problem-wise scores:

(1) Option (E): 11 people
(2) Option (D): 6 people
(3) Option (C): 11 people
(4) Option (B): 12 people

More details below.

2. Solutions

(1) Consider the expression x^2+t^2+xt. What is the derivative of this with respect to x (with t assumed to be a constant)?
(A) $2x+2t+x+t$
(B) $2x+2t+1$
(C) $2x+2t$
(D) $2x+t+1$
(E) $2x+t$

Answer: Option (E)

Explanation: When we differentiate with respect to x, keeping t constant, the x^2 differentiates to $2x$, the t^2 differentiates to 0 (because it is constant) and the xt differentiates to t.

Note that there's something else called *implicit differentiation* where we do not assume t to be a constant but rather an *unknown function of* x. In that case, we differentiate the functions of t with respect to t and tag along a dt/dx. With that interpretation, the derivative would be $2x+2t(dt/dx)+x(dt/dx)+t$. However, we're assuming t constant, so $dt/dx=0$, and so the dt/dx terms vanish and we are just left with $2t+x$.

The other choices:

Option (A) is what you get if you just differentiate each thing with respect to its own variable naively: x^2 gives $2x$, t^2 gives $2t$, xt, by the product rule, gives $x+t$. This is *completely wrong* because we are differentiating only with respect to x. (See the note on implicit differentiation right above this).

Options (B), (C) and (D) are also possible results of incorrect differentiation.

Performance review: 11 out of 12 people got this correct. 1 person chose (B).

Historical note (last year): 11 out of 12 people got this correct. 1 person chose (B).

History repeats itself!

(2) Which of the following verbal statements is **not valid as a general rule**?
(A) The derivative of the sum of two functions is the sum of the derivatives of the functions.
(B) The derivative of the difference of two functions is the difference of the derivatives of the functions.
(C) The derivative of a constant times a function is the same constant times the derivative of the function.

(D) The derivative of the product of two functions is the product of the derivatives of the functions.
(E) None of the above, i.e., they are all valid as general rules.

Answer: Option (D)

Explanation: The correct replacement of option (D) is the product rule for derivatives, which, in words, states that: "the derivative of the product of two functions is the sum of the product of the derivative of the first function with the second function and the product of the first function with the derivative of the second function." If that seems cumbersome to you, feel grateful for the power of algebraic symbols to capture this compactly:

$$(f \cdot g)' = (f' \cdot g) + (f \cdot g')$$

Performance review: 6 out of 12 people got this correct. 4 chose (E) and 2 chose (C).

Historical note (last year): 10 ouf to 12 people got this correct. 1 person chose (C) and 1 person chose (E).

Note on better performance last year: I think I got time to review the product rule for differentiation *before* the quiz last year, which was probably helpful.

(3) Which of the following statements is **definitely true** about the tangent line to the graph of an everywhere differentiable function f on $\mathbb{R}$ at the point $(a, f(a))$ (Here, "everywhere differentiable" means that the derivative of f is defined and finite for all $x \in \mathbb{R}$)?

(A) The tangent line intersects the curve at precisely one point, namely $(a, f(a))$.
(B) The tangent line intersects the x-axis.
(C) The tangent line intersects the $f(x)$-axis (the y-axis).
(D) Any line through $(a, f(a))$ other than the tangent line intersects the graph of f at at least one other point.
(E) None of the above need be true.

Answer: Option (C)

Explanation: If the function is differentiable, then the tangent line has finite slope, and hence cannot be vertical. Thus, it is not parallel to the y-axis, and hence must intersect the y-axis.

The other choices:

Option (A) is not true, as discussed in class. For instance, for the sin function, the tangent line through any of the peak points is $y = 1$, and passes through all the peak points, hence it intersects the graph infinitely often. We can graphically construct a lot of examples where the tangent line at one point in the graph intersects the graph elsewhere. There are certain classes of functions for which the statement of option (A) *is* true, and we'll talk more about this when we discuss concave up and concave down.

Option (B) is not true. The tangent line to $(\pi/2, 1)$ for the sin function is $y = 1$ – a horizontal line. Thus, it does not intersect the x-axis. In general, the tangent line does not intersect the x-axis iff it is horizontal, which happens iff the derivative at the point is zero.

Performance review: 11 out of 12 people got this correct. 1 chose (E).

Historical note (last year): 6 out of 12 people got this correct. 5 people chose option (E) (in other words, they were not convinced by any of the options (A)-(D)) and 1 person chose option (D).

Note on better performance this year: The question was starred and hence people were able to convince each other of the correct ideas through discussion.

(4) (*) For a function $f : (0, \infty) \to (0, \infty)$, denote by $f^{(k)}$ the k^{th} derivative of f. Suppose $f(x) := x^r$ with domain $(0, \infty)$, and r a rational number. What is the **precise set of values** of r satisfying the following: there exist a positive integer k (dependent on r) for which $f^{(k)}$ is identically the zero function.

(A) r should be an integer.
(B) r should be a nonnegative integer.
(C) r should be a positive integer.
(D) r should be a nonnegative rational number.
(E) r should be a positive rational number.

Answer: Option (B)

Explanation: If r is 0, then the derivative of the function is zero. For r a positive integer, the $(r+1)^{th}$ derivative is 0. See also Routine Problem 14 of Homework 3 (Exercise 3.3.64 of the book, Page 129).

For any other value of r, the power of x keeps going down by 1 each time we differentiate. However, since we didn't start with a nonnegative integer, the power of x never becomes 0, so we keep going down and never stop. For instance, if $r = 5/3$, we have:

$$f(x) = x^{5/3}, f^{(1)}(x) = (5/3)x^{2/3}, f^{(2)}(x) = (10/9)x^{-1/3}, \ldots$$

Note that the powers of x in f and its derivatives are 5/3, 2/3, $-1/3$, $-4/3$ and so on, going down by 1 each time. Note that going down from 2/3 to $-1/3$, the power skips right past 0.

Future teaser: This idea will come back at us with a vengeance when we study a technique called integration by parts in Math 153. For those who're brave enough, here's the key idea: one application of integration by parts is to integrate functions of the form (polynomial) times (function easy to repeatedly integrate). The main reason why this technique works for (polynomial)s is that if we differentiate a polynomial often enough, it becomes zero. If we try to apply the same idea replacing a polynomial by a power function such as $x^{5/3}$ (e.g., try $\int x^{5/3} \sin x \, dx$) then this does not work precisely because when repeatedly differentiating $x^{5/3}$, we skip right past the zero exponent.

Performance review: Everybody got this correct!

Historical note (last year): 4 out of 12 people got this correct. 2 people each chose (A), (C), (D) and (E).

Note on better performance this year: The question was starred allowing discussion, so people had the opportunity to convince each other of the right ideas. Nonetheless, everybody getting this correct is reasonably impressive, particularly considering that we haven't talked about these ideas in class so far.

CLASS QUIZ SOLUTIONS: OCTOBER 12: DERIVATIVES

MATH 152, SECTION 55 (VIPUL NAIK)

1. Performance review

12 people took this quiz. The score distribution is as follows:

- Score of 1: 6 people.
- Score of 2: 5 people.
- Score of 3: 1 person.

The mean score was 1.58.
Here are the problem-wise answers and scores:

(1) Option (C): 0 people
(2) Option (A): 4 people
(3) Option (E): 12 people. *Everybody got this correct!*
(4) Option (C): 3 people

2. Solutions

(1) (**) Suppose f is a differentiable function on $\mathbb{R}$. Which of the following implications is **false**?
(A) If f is even, then f' is odd.
(B) If f is odd, then f' is even.
(C) If f' is even, then f is odd.
(D) If f' is odd, then f is even.
(E) None of the above, i.e., they are all true.

Answer: Option (C)

Explanation: The function $f(x) := 3x + 1$ has derivative $f'(x) = 3$, which is even, but the original function f is not odd.

The key idea is that being an odd function has an additional condition, namely, that $f(0) = 0$, and the derivative provides no control over the value at a point, because we ca nadd a constant to a function and still retain the same derivative.

Performance review: *Nobody* got this correct. 7 chose (E), 3 chose (A), 1 chose (B), 1 chose (D).

Historical note (last year): *Nobody* got this correct. 11 people chose (E), 2 people chose (A), and 1 person chose (D).

Action point: This is a tricky problem. The reason why you were all led astray is that you were simply using examples, but did not have a wide enough repertoire of examples. You needed to think of examples of functions f where $f(0) \neq 0$ – examples as given above. Alternatively, you can try to do the theoretical derivation and stumble on the key insight that way.

The problem will probably become easier to think about when we reach indefinite integration.

(2) (*) A function f on $\mathbb{R}$ is said to satisfy the *intermediate value property* if, for any $a < b \in \mathbb{R}$, and any d between $f(a)$ and $f(b)$, there exists $c \in [a, b]$ such that $f(c) = d$. Which (one or more) of the following functions satisfies the intermediate value property?
(A) $f(x) := \{ \begin{array}{rl} \sin(1/x), & x \neq 0 \\ 0, & x = 0 \end{array}$
(B) $f(x) := \{ \begin{array}{ll} 1, & x \text{ rational} \\ 0, & x \text{ irrational} \end{array}$
(C) $f(x) := \{ \begin{array}{ll} x, & x \text{ rational} \\ 0, & x \text{ irrational} \end{array}$
(D) All of the above

(E) None of the above

Answer: Option (A)

Explanation: The crucial thing we use is that the range of f is $[-1, 1]$, and it takes all values in the range on any nonempty open interval. From this, it is easy to see that f satisfies the intermediate value property.

The other choices:

Option (B): This is not correct. The function takes the values 0 and 1 only. In particular, it does not take any of the intermediate values between 0 and 1.

Option (C): This is not correct. For instance, suppose $a = \sqrt{2}$ and $b = 1.44$. Then $f(a) = 0$ and $f(b) = 1.44$. The value 0.264 lies between $f(a)$ and $f(b)$. But there is nothing between a and b to which applying f gives 0.264.

Remark: The intermediate value theorem can be interpreted as the statement that any continuous function (on an interval) satisfies the intermediate value property. The example (A), however, illustrates that the converse to the intermediate value theorem does not hold, i.e., that there are functions that satisfy the intermediate value property but are not continuous.

Performance review: 4 out of 12 people got this correct. 4 chose (D), 3 chose (E), 1 chose (C).

Historical note (last year): 7 out of 14 people got this correct. 4 people chose (D) and 3 people chose (E). It is likely that the people who chose (D) first tried and found that (A) works, and assumed (incorrectly) that the other options work similarly.

(3) Which (one or more) of the following functions have a period of π?

(A) $x \mapsto \sin^2 x$

(B) $x \mapsto |\sin x|$

(C) $x \mapsto \cos^2 x$

(D) $x \mapsto |\cos x|$

(E) All of the above

Answer: Option (E)

Explanation: We have $\sin(x + \pi) = -\sin x$ and $\cos(x + \pi) = \cos x$. Thus, when we square or take the absolute value, we see that the function value repeats after an interval of π. It is also clear from the graph or by inspection that no smaller thing works as the period.

Performance review: *Everybody* got this correct.

Historical note (last year): 12 out of 14 people got this correct. 1 person chose (B) and (D) and 1 person chose (A).

(4) Suppose f is a function defined on all of $\mathbb{R}$ such that f' is a periodic function defined on all of $\mathbb{R}$. What can we conclude is **definitely true** about f?

(A) f must be a linear function.

(B) f must be a periodic function.

(C) f can be expressed as the sum of a linear and a periodic function, but f need not be either linear or periodic.

(D) f can be expressed as the product of a linear and periodic function, but f need not be either linear or periodic.

(E) f can be expressed as a composite of a linear and a periodic function, but f need not be either linear or periodic.

Answer: Option (C)

Explanation: The function $x + \sin x$ provides an example for option (C) that does not satisfy any of the descriptions of the other options. Thus, by eliminating other choices, we see that (C) is correct. A more formal explanation of why (C) works will have to wait till we reach the material leading up to indefinite integration.

Remark: Graphically, a sum of a linear function and a periodic function has a graph that repeats itself, but shifted over both vertically and horizontally. We'll talk quite a bit about such functions when we study graphing techniques. Another way of thinking of it is that the linear function represents the *secular trend* and the periodic function represents the *seasonal variation*. For instance, a graph of the daily sales revenue at a supermarket will have a secular trend (increasing, if the supermarket and its customer base are expanding over time, and decreasing if the customer base is shrinking) and a seasonal variation (spikes during Black Friday and Christmas season, lows at some times of the year). If the secular trend is linear, and the secular and seasonal trend interact additively, then the sales revenues are approximately represented as the sum of a linear and a periodic function. If they interact multiplicatively, then the sales revenues are approximately represented as the product of a linear and a periodic function.

Performance review: 3 out of 12 people got this correct. 9 chose (B).

Historical note (last year): 8 out of 14 people got this correct. 5 people chose (B), and 1 person chose (E).

Action point: It *is* true that if f is a periodic differentiable function, then f' is also periodic. However, that's not what the question is asking for. The question is asking for the reverse: if f' is a periodic function, what can we conclude about f? Is f also periodic? In fact, as the answer above makes clear, it is possible for a non-periodic function to have a periodic derivative.

CLASS QUIZ SOLUTIONS: OCTOBER 14: DERIVATIVES

MATH 152, SECTION 55 (VIPUL NAIK)

1. Performance review

11 people took this quiz. Everybody got all questions correct!
The problem wise answers and performance review are below:

(1) Option (E): Everybody
(2) Option (C): Everybody
(3) Option (D): Everybody
(4) Option (D): Everybody

Good job!

2. Solutions

(1) Suppose f and g are functions from $\mathbb{R}$ to $\mathbb{R}$ that are everywhere differentiable. Which of the following functions is/are guaranteed to be everywhere differentiable?
(A) $f + g$
(B) $f - g$
(C) $f \cdot g$
(D) $f \circ g$
(E) All of the above

Answer: Option (E)

Explanation: In fact, we have explicit formulas for the derivatives of all of these in terms of the derivatives of f and g. We have $(f + g)' = f' + g'$ and $(f - g)' = f' - g'$. We also have the product rule and chain rule for options (C) and (D).

Note that for the composition, we are using something more: since these are functions on the whole real line $\mathbb{R}$, the value $g(x)$ also lies in the domain of f, hence it makes sense to compose.

Performance review: Everybody got this correct

Historical note (last year): 13 out of 14 people got this correct. 1 person chose a multitude of options. *Note: The answer should always be exactly one option.*

(2) Suppose f and g are both twice differentiable functions everywhere on $\mathbb{R}$. Which of the following is the correct formula for $(f \cdot g)''$?
(A) $f'' \cdot g + f \cdot g''$
(B) $f'' \cdot g + f' \cdot g' + f \cdot g''$
(C) $f'' \cdot g + 2f' \cdot g' + f \cdot g''$
(D) $f'' \cdot g - f' \cdot g' + f \cdot g''$
(E) $f'' \cdot g - 2f' \cdot g' + f \cdot g''$

Answer: Option (C)

Explanation: We differentiate once to get:

$$(f \cdot g)' = f' \cdot g + f \cdot g'$$

Now we differentiate both sides. The left side becomes $(f \cdot g)''$. The right side is a sum of two terms, so we get:

$$(f \cdot g)'' = (f' \cdot g)' + (f \cdot g')'$$

We now apply the product rule to each piece on the right side to get:

$$(f \cdot g)'' = [f'' \cdot g + f' \cdot g'] + [f' \cdot g' + f \cdot g'']$$

Combining terms, we get option (C).

Remark: In general, there is a binomial theorem-like formula for the n^{th} derivative of $f \cdot g$. I've given the formula below, but it will make sense only to people who have seen summation notation and the binomial coefficients, which we have not yet done:

$$(f \cdot g)^{(n)} = \sum_{k=0}^{n} \binom{n}{k} f^{(k)} g^{(n-k)}$$

This is a lot like the binomial theorem expansion for $(a+b)^n$. It can be proved purely formally using induction from the product rule.[1]

Performance review: Everybody got this correct

Historical note (last year): 13 out of 14 people got this correct. 1 person chose option (E), though that person's rough work gave option (C).

(3) Suppose f and g are both twice differentiable functions everywhere on $\mathbb{R}$. Which of the following is the correct formula for $(f \circ g)''$?

(A) $(f'' \circ g) \cdot g''$
(B) $(f'' \circ g) \cdot (f' \circ g') \cdot g''$
(C) $(f'' \circ g) \cdot (f' \circ g') \cdot (f \circ g'')$
(D) $(f'' \circ g) \cdot (g')^2 + (f' \circ g) \cdot g''$
(E) $(f' \circ g') \cdot (f \circ g) + (f'' \circ g'')$

Answer: Option (D)

Explanation: This question is tricky because it requires the application of both the product rule and the chain rule, with the latter being used twice. We first note that:

$$(f \circ g)' = (f' \circ g) \cdot g'$$

Now, we differentiate both sides:

$$(f \circ g)'' = [(f' \circ g) \cdot g']'$$

The expression on the right side that needs to be differentiated is a product, so we use the product rule:

$$(f \circ g)'' = [(f' \circ g)' \cdot g'] + [(f' \circ g) \cdot g'']$$

Now, the inner composition $f' \circ g$ needs to be differentiated. We use the chain rule and obtain that $(f' \circ g)' = (f'' \circ g) \cdot g'$. Plugging this back in, we get:

$$(f \circ g)'' = (f'' \circ g) \cdot (g')^2 + (f' \circ g) \cdot g''$$

Remark: What's worth noting here is that in order to differentiate composites of functions, you need to use both composites *and* products (that's the chain rule). And in order to differentiate products, you need to use both products *and* sums (that's the product rule). Thus, in order to differentiate a composite twice, we need to use composites, products, *and* sums.

Performance review: Everybody got this correct

Historical note (last year): 14 out of 14 people got this correct. This is great! I had expected that many of you would be put off by the messy computation, but apparently you were unfazed.

(4) Suppose f is an everywhere differentiable function on $\mathbb{R}$ and $g(x) := f(x^3)$. What is $g'(x)$?

(A) $3x^2 f(x)$
(B) $3x^2 f'(x)$
(C) $3x^2 f(x^3)$
(D) $3x^2 f'(x^3)$

[1] One of the things I'm doing research on has to do with the fact above, albeit with a completely different notion of differentiation.

(E) $f'(3x^2)$

Answer: Option (D)

Explanation: Put $h(x) := x^3$. Then $g = f \circ h$. Thus, $g'(x) = f'(h(x))h'(x) = f'(x^3) \cdot (3x^2)$, giving option (D).

Performance review: Everybody got this correct

Historical note (last year): 13 out of 14 people got this correct. 1 person chose option (B), which is a close distractor if you're not paying attention.

CLASS QUIZ SOLUTIONS: OCTOBER 19: INCREASE/DECREASE AND MAXIMA/MINIMA

MATH 152, SECTION 55 (VIPUL NAIK)

1. Performance review

11 people took the quiz. The score distribution was as follows:

- Score of 2: 8 people
- Score of 3: 3 people

The mean score was 2.08. The problem wise performance was:

(1) Option (D): 10 people
(2) Option (B): 6 people
(3) Option (B): 2 people
(4) Option (B): 7 people

2. Solutions

(1) Suppose f is a function defined on a closed interval $[a, c]$. Suppose that the left-hand derivative of f at c exists and equals ℓ. Which of the following implications is **true in general**?
 (A) If $f(x) < f(c)$ for all $a \le x < c$, then $\ell < 0$.
 (B) If $f(x) \le f(c)$ for all $a \le x < c$, then $\ell \le 0$.
 (C) If $f(x) < f(c)$ for all $a \le x < c$, then $\ell > 0$.
 (D) If $f(x) \le f(c)$ for all $a \le x < c$, then $\ell \ge 0$.
 (E) None of the above is true in general.

 Answer: Option (D)

 Explanation: If $f(x) \le f(c)$ for all $a \le x < c$, then all difference quotients from the left are nonnegative. The limiting value, which is the left-hand derivative, is thus also nonnegative. See the lecture notes for more details.

 The other choices: Options (A) and (B) predict the wrong sign. Option (C) is incorrect because even though the difference quotients are all strictly positive, their limiting value could be 0. For instance, $\sin x$ on $[0, \pi/2]$ or x^3 on $[-1, 0]$.

 Performance review: 10 out of 11 got this correct. 1 person chose (E).

 Historical note (last year): 8 people got this correct. 5 people chose option (B) and 2 people chose option (E). It is likely that the people who chose option (B) made a sign computation error.

 Action point: On the plus side, most of you seem to have understood the fact that strict inequality does not guarantee strict positivity or negativity of the one-sided derivative. But, please sort out your sign issues while the quarter is still young! Getting the right sign is a good sign for the future.

(2) Suppose f and g are increasing functions from $\mathbb{R}$ to $\mathbb{R}$. Which of the following functions is *not* guaranteed to be an increasing function from $\mathbb{R}$ to $\mathbb{R}$?
 (A) $f + g$
 (B) $f \cdot g$
 (C) $f \circ g$
 (D) All of the above, i.e., none of them is guaranteed to be increasing.
 (E) None of the above, i.e., they are all guaranteed to be increasing.

 Answer: Option (B)

 Explanation: The problem with option (B) arises when one or both functions take negative values. For instance, consider the case $f(x) := x$ and $g(x) := x$. Both are increasing functions on all of $\mathbb{R}$.

However, the pointwise product is the function $x \mapsto x^2$, which is a decreasing function for negative x.

Formally, the issue is that we cannot multiply inequalities of the form $A < B$ and $C < D$ unless we are guaranteed to be working with positive numbers.

The other choices:

Option (A): For any $x_1 < x_2$, we have $f(x_1) < f(x_2)$ and $g(x_1) < g(x_2)$. Adding up, we get $f(x_1) + g(x_1) < f(x_2) + g(x_2)$, so $(f + g)(x_1) < (f + g)(x_2)$.

Option (C): For any $x_1 < x_2$, we have $g(x_1) < g(x_2)$ since g is increasing. Now, we use the factthat f is increasing to compare its values at the two points $g(x_1)$ and $g(x_2)$, and we get $f(g(x_1)) < f(g(x_2))$. We thus get $(f \circ g)(x_1) < (f \circ g)(x_2)$.

Performance review: 6 out of 11 got this correct. 2 chose (C) and 3 chose (E).

Historical note (last year): Only 1 person got this correct! 8 people chose option (E), 4 people chose option (C), 1 person chose option (D), and 1 person chose (A)+(B). Note that you'll always have exactly one correct answer option.

From the rough work shown by a few people, it seems that a lot of people were trying to reason this problem using derivatives. Using derivatives is *not* a sound approach to tackling this problem because it is not given that the function is differentiable or even continuous. Nonetheless, it is possible to obtain the correct answer using the flawed approach of derivatives, and it is sad that so few of you did so.

Others seem to have used examples. With examples, you should have found the counterexample rather easily, if you'd chosen $f(x) = g(x) = x$. However, most of you don't seem to have considered a sufficiently wide range of examples and to have settled with a few random ones. This is *not* the right way to use examples. When searching for counterexamples, you should look systematically and try to vary the essential features in a meaningful manner. More on this if we get time to cover this material in problem session.

Action point: Please, please make sure you understand this kind of problem so well that in the future, you're puzzled that this ever confused you. Unlike formulas for differentiating complicated functions, which you may forget a few years after doing calculus, the reasoning methods for these kinds of questions should stick with you for a lifetime.

(3) Suppose f is a continuous function defined on an open interval (a, b) and c is a point in (a, b). Which of the following implications is **true**?
(A) If c is a point of local minimum for f, then there is a value $\delta > 0$ and an open interval $(c - \delta, c + \delta) \subseteq (a, b)$ such that f is non-increasing on $(c - \delta, c)$ and non-decreasing on $(c, c + \delta)$.
(B) If there is a value $\delta > 0$ and an open interval $(c - \delta, c + \delta) \subseteq (a, b)$ such that f is non-increasing on $(c - \delta, c)$ and non-decreasing on $(c, c + \delta)$, then c is a point of local minimum for f.
(C) If c is a point of local minimum for f, then there is a value $\delta > 0$ and an open interval $(c - \delta, c + \delta) \subseteq (a, b)$ such that f is non-decreasing on $(c - \delta, c)$ and non-increasing on $(c, c + \delta)$.
(D) If there is a value $\delta > 0$ and an open interval $(c - \delta, c + \delta) \subseteq (a, b)$ such that f is non-decreasing on $(c - \delta, c)$ and non-increasing on $(c, c + \delta)$, then c is a point of local minimum for f.
(E) All of the above are true.

Answer: Option (B).

Explanation: Since f is continuous, being non-increasing on $(c - \delta, c)$ implies being non-increasing on $(c - \delta, c]$. Similarly on the right side. In particular, this means that $f(c) \le f(x)$ for all $x \in (c - \delta, c + \delta)$, establishing c as a point of local minimum.

The other choices: Options (C) and (D) have the wrong kind of increase/decrease. Option (A) is wrong, though counterexamples are hard to come by. The reason Option (A) is wrong is the core of the reason that the first-derivative test does not always work: the function could be oscillatory very close to the point c, so that even though c is a point of local minimum, the function does not steadily become non-increasing to the left of c. The example discussed in the lecture notes is $|x|(2+\sin(1/x))$.

Performance review: 2 out of 11 got this. 6 chose (A) and 1 each chose (C), (D), and (E).

Historical note (last year): 5 people got this correct. 5 people chose (A), which is the converse of the statement. 2 people chose (D) and 1 person each chose (C) and (E). Thus, most people got the sign/direction part correct but messed up on which way the implication goes.

Action point: This is tricky to get when the lecture material is still very new to you. However, you should consistently get this kind of question correct once you have reviewed and mastered the lecture material.

(4) Suppose f is a continuously differentiable function on $\mathbb{R}$ and f' is a periodic function with period h. (Recall that periodic derivative implies that the original function is a sum of ...). Suppose S is the set of points of local maximum for f, and T is the set of local maximum values. Which of the following is **true in general** about the sets S and T?

(A) The set S is invariant under translation by h (i.e., $x \in S$ if and only if $x + h \in S$) and all the values in the set T are in the image of the set $[0, h]$ under f.

(B) The set S is invariant under translation by h (i.e., $x \in S$ if and only if $x + h \in S$) but all the values in the set T need not be in the image of the set $[0, h]$ under f.

(C) Both the sets S and T are invariant under translation by h.

(D) Both the sets S and T are finite.

(E) Both the sets S and T are infinite.

Answer: Option (B).

Explanation: For a differentiable function, whether a point is a point of local maximum or not depends only on the derivative behavior near the point. Since the derivative is periodic with period h, S is invariant under translation by h.

The set of values T may be finite or infinite. If f itself is periodic, then the set of maximum values over a single period is the same as the set of maximum values overall. Thus, for instance, in the case of the function $f(x) := \sin x$, $T = \{1\}$ and $S = \{\pi/2 + 2n\pi : n \in \mathbb{Z}\}$. On the other hand, for the function $f(x) := 2\sin x - x$, the derivative is $f'(x) = 2\cos x - 1$, which is zero at $2n\pi \pm \pi/3$. The local maxima are attained at $2n\pi + \pi/3$ (as we can see from either derivative test). However, the set of local maximum *values* is infinite – in fact, each n gives a different local maximum value, and the set of local maximum values is infinite and unbounded. For instance, for $n = 0$, the local maximum value is $f(\pi/3) = \sqrt{3} - \pi/3$. For $n = 1$, the local maximum value is $f(7\pi/3) = \sqrt{3} - (7\pi/3)$. And so on.

Performance review: 7 out of 11 got this correct. 3 chose (E) and 1 chose (C).

Historical note (last year): 3 people got this correct. 3 people each chose (A) and (D), 4 people chose (C), and 2 people chose (E).

Action point: This was a fairly hard problem. However, it should become easy after we've seen a little more about functions with periodic derivative.

CLASS QUIZ SOLUTIONS: OCTOBER 21: MAX-MIN PROBLEMS

MATH 152, SECTION 55 (VIPUL NAIK)

1. Performance review

Note: The quiz was submitted in class on October 26.
12 students took this quiz. The score distribution was as follows:

- Score of 3: 2 persons.
- Score of 4: 5 persons.
- Score of 6: 4 persons.
- Score of 7: 1 person.

The mean score was 4.75. Here are the problem solutions and problem wise scores:

(1) Option (B): Everybody
(2) Option (B): Everybody
(3) Option (D): Everybody
(4) Option (A): 6 people
(5) Option (A): 6 people
(6) Option (C): 4 people
(7) Option (C): 5 people

2. Solutions

(1) Consider all the rectangles with perimeter equal to a fixed length $p > 0$. Which of the following **is true** for the unique rectangle which is a square, compared to the other rectangles?
(A) It has the largest area and the largest length of diagonal.
(B) It has the largest area and the smallest length of diagonal.
(C) It has the smallest area and the largest length of diagonal.
(D) It has the smallest area and the smallest length of diagonal.
(E) None of the above.

Answer: Option (B)

Explanation: We can see this easily by doing calculus, but it can also be deduced purely by thinking about how a square and a long thin rectangle of the same perimeter compare in terms of area and diagonal length.

Performance review: Everybody got this correct.

Historical note (last year): Everybody got this correct.

Historical note (two years ago): This question appeared on last year's 151 final, and 31 out of 33 people got it correct.

(2) Suppose the total perimeter of a square and an equilateral triangle is L. (We can choose to allocate all of L to the square, in which case the equilateral triangle has side zero, and we can choose to allocate all of L to the equilateral triangle, in which case the square has side zero). Which of the following statements **is true** for the sum of the areas of the square and the equilateral triangle? (The area of an equilateral triangle is $\sqrt{3}/4$ times the square of the length of its side).
(A) The sum is minimum when all of L is allocated to the square.
(B) The sum is maximum when all of L is allocated to the square.
(C) The sum is minimum when all of L is allocated to the equilateral triangle.
(D) The sum is maximum when all of L is allocated to the equilateral triangle.
(E) None of the above.

Answer: Option (B)

Quick explanation: The problem can also be solved using the rough heuristic that works for these kinds of problems: the maximum occurs when everything is allocated to the most efficient use, but the minimum typically occurs somewhere in between.

Full explanation: Suppose x is the part allocated to the square. Then $L - x$ is the part allocated to the equilateral triangle. The total area is:

$$A(x) = x^2/16 + (\sqrt{3}/4)(L - x)^2/9$$

Differentiating, we obtain:

$$A'(x) = \frac{x}{8} - \frac{\sqrt{3}}{18}(L - x) = x\left(\frac{1}{8} + \frac{\sqrt{3}}{18}\right) - \frac{\sqrt{3}}{18}L$$

We see that $A'(x) = 0$ at

$$x = \frac{L(\sqrt{3}/18)}{(1/8) + (\sqrt{3}/18)}$$

This number is indeed within the range of possible values of x.

Further, $A'(x) > 0$ for x greater than this and $A'(x) < 0$ for x less than this. Thus, this point is a local minimum and the maximum must occur at one of the endpoints. We plug in $x = 0$ to get $(\sqrt{3}/36)L^2$ and we plug in $x = L$ to get $L^2/16$. Since $1/16 > \sqrt{3}/36$, we obtain the the maximum occurs when $x = L$, which means that all the perimeter goes to the square.

Performance review: Everybody got this correct.

Historical note (last year): 9 out of 15 people got this correct. 2 people each chose (E) and (C), 1 person chose (D), and 1 person left the question blank.

Historical note: This question appeared in a 152 midterm last year, and 20 of 29 people got it correct. This is a somewhat better showing than you lot, but that midterm occurred after several homeworks, lectures, and two review sessions covering max-min problems. Also, in that midterm, option (E) wasn't there, so things became a little easier.

(3) Suppose x and y are positive numbers such as $x + y = 12$. For **what values** of x and y is x^2y maximum?
(A) $x = 3$, $y = 9$
(B) $x = 4$, $y = 8$
(C) $x = 6$, $y = 6$
(D) $x = 8$, $y = 4$
(E) $x = 9$, $y = 3$

Answer: Option (D).

Quick explanation: This is a special case of the general Cobb-Douglas situation where we want to maximize $x^a(C - x)^b$. The general solution is to take $x = Ca/(a + b)$, i.e., to take x and $C - x$ in the proportion of a to b.

Full explanation: We need to maximize $f(x) := x^2(12-x)$, subject to $0 < x < 12$. Differentiating, we get $f'(x) = 3x(8 - x)$, so 8 is a critical point. Further, we see that f' is positive on $(0, 8)$ and negative on $(8, 12)$, so f attains its maximum (in the interval $(0, 12)$) at 8.

Performance review: Everybody got this correct.

Historical note (last year): 12 out of 15 people got this correct. 2 people chose (E) and 1 person chose (B). Of the people who got this correct, some seem to have computed the numerical values and others seem to have used calculus. Some who did not show any work may have used the general result of the Cobb-Douglas situation.

(4) Consider the function $p(x) := x^2 + bx + c$, with x restricted to integer inputs. Suppose b and c are integers. The minimum value of p is attained either at a single integer or at two consecutive integers. Which of the following is a **sufficient condition** for the minimum to occur at two consecutive integers?
(A) b is odd

(B) b is even
(C) c is odd
(D) c is even
(E) None of these conditions is sufficient.

Answer: Option (A)

Explanation: The graph of f is symmetric about the half-integer axis value $-b/2$. It is an upward-facing parabola. For odd b, it attains its minimum among integers at the two consecutive integers $-b/2 + 1/2$ and $-b/2 - 1/2$. When b is even, the minimum is attained uniquely at $-b/2$, which is itself an integer. c being odd or even tells us nothing.

Performance review: 6 out of 12 got this correct. 5 chose (E), 1 chose (B).

Historical note (last year): 4 out of 15 people got this correct. 8 people chose (E) and 3 people chose (B).

Action point: While this problem can be solved using calculus, it is much easier if you already know and remember important facts about the graphs of quadratic functions. Please review basic facts about quadratic functions.

(5) Consider a hollow cylinder with no top and bottom and total curved surface area S. What can we say about the **maximum and minimum** possible values of the **volume**? (for radius r and height h, the curved surface area is $2\pi rh$ and the volume is $\pi r^2 h$).
(A) The volume can be made arbitrarily small (i.e., as close to zero as we desire) and arbitrarily large (i.e., as large as we want).
(B) There is a positive minimum value for the volume, but it can be made arbitrarily large.
(C) There is a finite maximum value for the volume, but it can be made arbitrarily small.
(D) There is both a finite positive minimum and a finite positive maximum for the volume.

Answer: Option (A)

Explanation: We have $h = S/2\pi r$. Thus, the volume is $rS/2$. We see that as $r \to \infty$, the volume goes to infinity, and as $r \to 0$, the volume tends to zero. Thus, the volume can be made arbitrarily large as well as arbitrarily small.

Performance review: 6 out of 12 got this correct. 4 chose (C), 2 chose (D).

Historical note (last year): 6 out of 15 people got this correct. 3 people chose (B) and 6 people chose (C).

Action point: Please make sure you understand this problem, and also how it differs in nature from the next two problems.

(6) Consider a hollow cylinder with a bottom but no top and total surface area (curved surface plus bottom) S. What can we say about the **maximum and minimum** possible values of the **volume**? (for radius r and height h, the curved surface area is $2\pi rh$ and the volume is $\pi r^2 h$).
(A) The volume can be made arbitrarily small (i.e., as close to zero as we desire) and arbitrarily large (i.e., as large as we want).
(B) There is a positive minimum value for the volume, but it can be made arbitrarily large.
(C) There is a finite maximum value for the volume, but it can be made arbitrarily small.
(D) There is both a finite positive minimum and a finite positive maximum for the volume.

Answer: Option (C)

Quick explanation: We see that the radius cannot be expanded too much, otherwise the area of the bottom will itself exceed S. This puts a constraint on the total volume. On the other hand, the cylinder can be made arbitrarily thin and thus have arbitrarily small volume.

Full explanation: Try it yourself! There is a worked example in the book that essentially computes the maximum with specific numerical values – locate it!

Performance review: 4 out of 12 got this correct. 3 each chose (B) and (D), 2 chose (A).

Historical note (last year): 8 out of 15 people got this correct. 1 person chose (A), 4 people chose (B), and 2 people chose (D).

(7) Consider a hollow cylinder with a bottom and a top and total surface area (curved surface plus bottom and top) S. What can we say about the **maximum and minimum** possible values of the **volume**? (for radius r and height h, the curved surface area is $2\pi rh$ and the volume is $\pi r^2 h$).

(A) The volume can be made arbitrarily small (i.e., as close to zero as we desire) and arbitrarily large (i.e., as large as we want).
(B) There is a positive minimum value for the volume, but it can be made arbitrarily large.
(C) There is a finite maximum value for the volume, but it can be made arbitrarily small.
(D) There is both a finite positive minimum and a finite positive maximum for the volume.

Answer: Option (C)

Quick explanation: Identical to the previous problem.

Full explanation: Try it yourself!

Performance review: 5 out of 12 got this correct. 4 chose (D), 2 chose (B), and 1 chose (A).

Historical note (last year): 5 out of 15 people got this correct. 8 people chose (D) and 2 people chose (A).

Action point: Please try to understand, both conceptually and computationally, why the qualitative conclusion for this problem is the same as for the previous problem.

CLASS QUIZ SOLUTIONS: OCTOBER 24: CONCAVE, INFLECTIONS, TANGENTS, CUSPS, ASYMPTOTES

MATH 152, SECTION 55 (VIPUL NAIK)

1. Performance review

12 students took the quiz. The score distribution was as follows:

- Score of 2: 2 people
- Score of 3: 3 people
- Score of 4: 5 people
- Score of 5: 2 people

The mean score was 3.6.
Here are the problem wise answers:

(1) Option (C): 9 people
(2) Option (A): 5 people
(3) Option (C): 6 people
(4) Option (C): 11 people
(5) Option (B): 3 people
(6) Option (A): 6 people
(7) Option (E): 3 people

2. Solutions

(1) Consider the function $f(x) := x^3(x-1)^4(x-2)^2$. Which of the following **is true**?
(A) 0, 1, and 2 are all critical points and all of them are points of local extrema.
(B) 0, 1, and 2 are all critical points, but only 0 is a point of local extremum.
(C) 0, 1, and 2 are all critical points, but only 1 and 2 are points of local extrema.
(D) 0, 1, and 2 are all critical points, and none of them is a point of local extremum.
(E) 1 and 2 are the only critical points.

Answer: Option (C)

Quick explanation: 0, 1, and 2 are all roots of f with multiplicity greater than one, hence they are also roots of the derivative. Moreover, their multiplicity in the derivative is one less than their multiplicity in the original function.

To see which ones are local extrema, just think of the functions x^3, $(x-1)^4$, and $(x-2)^2$ as isolated functions.

Alternatively, since all these are critical points where the function is also zero, we can, instead of using the derivative test, directly compute the sign of the function to the left and the right of each critical points. For those critical points where there is an even power, the sign of the original function is the same on both sides close to the point. For thos ecritical points where there is an odd power, the sign of the original function flips.

Full explanation: Try it yourself! You need to calculate the first derivative, see where it is zero, etc.

Performance review: 9 out of 12 got this correct. 2 chose (B), 1 chose (D).

Historical note (last year): 11 out of 15 people got this correct, which is a good showing. Other choices were (B) (2), (A) (1), and (E) (1).

However, many people did tedious derivative computations for this question. Please try to understand the intuition behind how this problem can be solved without computing derivatives.

(2) Suppose f and g are continuously differentiable functions on $\mathbb{R}$. Suppose f and g are both concave up. Which of the following is **always true**?
(A) $f + g$ is concave up.
(B) $f - g$ is concave up.
(C) $f \cdot g$ is concave up.
(D) $f \circ g$ is concave up.
(E) All of the above.

Answer: Option (A)

Explanation: The sum of two increasing functions is increasing. Hence, if f' and g' are both increasing, so is the sum $f' + g' = (f + g)'$. The other options are false. In fact, for any two functions that are concave up, both the differences $f - g$ and $g - f$ cannot be concave up. As for products, consider the example of $f(x) = x^2$ and $g(x) = (x - 1)^2$, which are both concave up everywhere, but their product is not. As for composites, consider $f(x) = x^2$ and $g(x) = x^2 - 2x$. The composite is $(x^2 - 2x)^2$, which is not concave up everywhere.

Performance review: 5 out of 12 got this correct. 3 chose (D), 2 each chose (C) and (E).

Historical note (last year): 8 out of 15 people got this correct. Other choices were (E) (5), (C) (1), and (D) (1).

(3) Consider the function $p(x) := x(x - 1) \dots (x - n)$, where $n \geq 1$ is a positive integer. How many points of inflection does p have?
(A) $n - 3$
(B) $n - 2$
(C) $n - 1$
(D) n
(E) $n + 1$

Answer: Option (C)

Explanation: p has degree $n + 1$, since it is the product of $n + 1$ linear polynomials. Thus, p'' has degree $n - 1$, and hence can have at most $n - 1$ roots. Thus, the number of points of inflection of p is at most $n - 1$. If we can locate $n - 1$ points of inflection, we will be done.

Note that since p has zeros at $0, 1, 2, \dots, n$. By the extreme value theorem, there exists a local extreme value for p between any two consecutive zeros, and this gives a root of p'. By degree considerations, there must be exactly one root in each interval $(i - 1, i)$. There are thus n distinct roots of p', each giving a local extreme value of p, located in the intervals $(0, 1), (1, 2), \dots, (n - 1, n)$. Call these roots $a_1, a_2, \dots, a_n$. Then $a_1 < a_2 < \dots < a_n$. Applying the extreme value theorem again on the intervals $[a_{i-1}, a_i]$, we see that there is at least one local extremum for p', and hence a zero for p'', in that interval. Since p'' has degree $n - 1$, there must be exactly one local extremum on each interval. Since local extrema of the derivative correspond to points of inflection, we have found $n - 1$ distinct points of inflection, and we are done.

Performance review: 6 out of 12 got this correct. 3 each chose (B) and (E).

Historical note (last year): 7 out of 15 people got this correct. Other choices were (B) (3), (D) (3), (A) (1), and (E) (1). Those people who chose (D) typically made the error of doing $(n + 1) - 1$ instead of $(n + 1) - 2$.

(4) Suppose f is a polynomial function of degree $n \geq 2$. What can you say about the sense of concavity of the function f for **large enough inputs**, i.e., as $x \to +\infty$? (Note that if $n \leq 1$, f is linear so we do not have concavity in either sense).
(A) f is eventually concave up.
(B) f is eventually concave down.
(C) f is eventually either concave up or concave down, and which of these cases occurs depends on the sign of the leading coefficient of f.
(D) f is eventually either concave up or concave down, and which of these cases occurs depends on whether the degree of f is even or odd.
(E) f may be concave up, concave down, or neither.

Apologies, the language of option (E) was confusing. Although the question did say "eventually", option (E) should ideally have read "f may be eventually concave up, concave down, or neither" rather than forcing you to infer this from the context.

Answer: Option (C).

Explanation: Note first that the sign of the leading coefficient of f'' is the same as the sign of the leading coefficient of f, because the leading coefficient gets multiplied by $n(n-1)$, which is positive for $n \geq 2$. If this sign is positive, then $f''(x) \to \infty$ as $x \to \infty$, and hence must be eventually positive, forcing f to be eventually concave up. If this sign is negative, then $f''(x) \to -\infty$ as $x \to \infty$, and hence must be eventually negative, forcing f to be eventually concave down.

Performance review: 11 out of 12 got this correct. 1 chose (D).

Historical note (last year): 12 out of 15 people got this correct. 2 people chose option (E). This is probably partly because of the confusing language, see the apology above.

(5) Suppose f is a continuously differentiable function on $[a, b]$ and f' is continuously differentiable at all points of $[a, b]$ except an interior point c, where it has a vertical cusp. What can we say is **definitely true** about the behavior of f at c?

(A) f attains a local extreme value at c.
(B) f has a point of inflection at c.
(C) f has a critical point at c that does not correspond to a local extreme value.
(D) f has a vertical tangent at c.
(E) f has a vertical cusp at c.

Answer: Option (B)

Explanation: A cusp for f' means that f' changes direction at c (either from increasing to decreasing or from decreasing to increasing). This in turn means that the sense of concavity of f changes at c. Hence, f has a point of inflection at c. Note that we cannot have the vertical tangent situation because f' is continuous and finite at c.

An example of such a function f would be $f(x) := x^{5/3}$. The first derivative $f'(x) = (5/3)x^{2/3}$ is everywhere defined and has a vertical cusp at $c = 0$. We note that the derivative switches from decreasing to increasing at $c = 0$, so the original function switches from concave down to concave up.

Performance review: 3 out of 12 got this correct. 6 chose (C), 2 chose (A), 1 chose (E).

Historical note (last year): 3 out of 15 people got this correct. Other options were (A) (4), (E) (3), (D) (3), and (C) (2). It seems that a lot of people did not make a clear enough distinction between the roles of f and f'.

Action point: This is the kind of question that is hard at first but at some stage should feel obvious to you. (Not immediately obvious, since it still requires you to read carefully, but the kind of thing that would not confuse you).

(6) Suppose f and g are continuous functions on $\mathbb{R}$, such that f attains a vertical tangent at a and is continuously differentiable everywhere else, and g attains a vertical tangent at b and is continuously differentiable everywhere else. Further, $a \neq b$. What can we say is **definitely true** about $f - g$?

(A) $f - g$ has vertical tangents at a and b.
(B) $f - g$ has a vertical tangent at a and a vertical cusp at b.
(C) $f - g$ has a vertical cusp at a and a vertical tangent at b.
(D) $f - g$ has no vertical tangents and no vertical cusps.
(E) $f - g$ has either a vertical tangent or a vertical cusp at the points a and b, but it is not possible to be more specific without further information.

Answer: Option (A)

Explanation: Note that $\lim_{x \to b}(f - g)'(x) = f'(b) - \lim_{x \to b} g'(x)$, which is an infinity of the sign opposite to that of g'. In particular, we hav a vertical tangent at b. Similarly, we have a vertical tangent at a.

Performance review: 6 out of 12 got this correct. 5 chose (E), 1 chose (B).

Historical note (last year): 5 out of 15 people got this correct. Other choices were (E) (5), (C) (3), (B) (1), (D) (1). The main source of confusion here seems to have been that people did not realize if f has a vertical tangent at c, so does $-f$, because the whole picture flips over.

(7) Suppose f and g are continuous functions on $\mathbb{R}$, such that f is continuously differentiable everywhere and g is continuously differentiable everywhere except at c, where it has a vertical tangent. What can we say is **definitely true** about $f \circ g$?
(A) It has a vertical tangent at c.
(B) It has a vertical cusp at c.
(C) It has either a vertical tangent or a vertical cusp at c.
(D) It has neither a vertical tangent nor a vertical cusp at c.
(E) We cannot say anything for certain.

Answer: Option (E).

Explanation: Consider $g(x) := x^{1/3}$. This has a vertical tangent at $c = 0$. If we choose $f(x) = x$, we get (A). If we choose $f(x) = x^2$, we get (B). If we choose $f(x) = x^3$, we get neither a vertical tangent nor a vertical cusp. Hence, (E) is the only viable option.

Performance review: 3 out of 12 got this correct. 4 chose (C), 2 each chose (A) and (D), 1 chose (B).

Historical note (last year): 3 out of 15 people got this correct. Other choices were (A) (7), (C) (4), and (D) (1). The main thing that people had trouble with was thinking of possibilities for f that could play the role of converting the vertical tangent behavior of the original function g into vertical cusp or "neither" behavior for the composite function.

Action point: This is a devilishly tricky question that I don't expect you to get at all in your first try, but that I expect that you will remember for the rest of your life (or at least this course) after you've seen it.

CLASS QUIZ SOLUTIONS: OCTOBER 28: INTEGRATION BASICS

MATH 152, SECTION 55 (VIPUL NAIK)

1. Performance review

This quiz actually happened on November 4.
12 people took this quiz. The score distribution was as follows:

- Score of 0: 3 people
- Score of 1: 2 people
- Score of 2: 4 people
- Score of 3: 2 people
- Score of 4: 1 person

The mean score was 1.67.
Here are the problem-wise answers and performance:

(1) Option (C): 4 people
(2) Option (D): 7 people
(3) Option (D): 4 people
(4) Option (A): 5 people

2. Solutions

(1) Consider the function(s) $[0,1] \to \mathbb{R}$. **Identify the function**s for which the integral (using upper sums and lower sums) is not defined.
   (A) $f_1(x) := \begin{cases} 0, & 0 \le x < 1/2 \\ 1, & 1/2 \le x \le 1 \end{cases}$.
   (B) $f_2(x) := \begin{cases} 0, & x \neq 0 \text{ and } 1/x \text{ is an integer} \\ 1, & \text{otherwise} \end{cases}$.
   (C) $f_3(x) := \begin{cases} 0, & x \text{ rational} \\ 1, & x \text{ irrational} \end{cases}$
   (D) All of the above
   (E) None of the above

   Answer: Option (C)

   Explanation: For option (C), the lower sum for any partition is 0 and the upper sum is 1. Thus, the integral is not well-defined.

   For option (A), the function is piecewise continuous with only jump discontinuities, hence the integral is well-defined: in fact, it is 1/2.

   For (B), the integral is zero. We can see this by noting that the points where the function is 0 are all isolated points, so if in our partition the intervals surrounding each of these points is small enough, we can make the upper sums tend to zero. (This is hard to see. You should, however, be able to easily see that (A) has an integral and (C) does not. This forces the answer to be (C)).

   Performance review: 4 out of 12 got this correct. 3 each chose (B) and (D), 2 chose (E).

   Historical note (last year): Everybody got this correct.

(2) (**) Suppose $a < b$. Recall that a *regular partition* into n parts of $[a, b]$ is a partition $a = x_0 < x_1 < \dots < x_{n-1} < x_n = b$ where $x_i - x_{i-1} = (b-a)/n$ for all $1 \le i \le n$. A partition P_1 is said to be a *finer partition* than a partition P_2 if the set of points of P_1 contains the set of points of P_2. Which of the following is a **necessary and sufficient condition** for the regular partition into m parts to be a *finer partition* than the regular partition into n parts? (Note: We'll assume that any partition is finer than itself).

(A) $m \leq n$
(B) $n \leq m$
(C) m divides n (i.e., n is a multiple of m)
(D) n divides m (i.e., m is a multiple of n)
(E) m is a power of n

Answer: Option (D)

Explanation: If n divides m, then the partition into m pieces is obtained by further subdividing the partition into m parts, with each part divided into n/m parts.

The other choices: Option (B) is a *necessary* condition but is not a sufficient condition. For instance, the regular partition of $[0, 1]$ into two parts corresponds to $\{0, 1/2, 1\}$ and the partition into three parts corresponds to $\{0, 1/3, 2/3, 1\}$. These partitions are incomparable, i.e., neither is finer than the other.

Performance review: 7 out of 12 got this correct. 4 chose(C), 1 chose (B).

Historical note (last year): 5 out of 15 people got this correct. 6 people chose (B), which is necessary but not sufficient, as indicated above. 2 people chose (C), which is the reverse option. 1 person chose (A) and 1 chose (A)+(D).

Action point: If you chose option (B), please make sure you understand the distinction between the options. Also review the concept of regular partitions and finer partition till you find this question obvious.

(3) (**) For a partition $P = x_0 < x_1 < x_2 < \cdots < x_n$ of $[a, b]$ (with $x_0 = a$, $x_n = b$) define the norm $\|P\|$ as the maximum of the values $x_i - x_{i-1}$. Which of the following **is always true** for any continuous function f on $[a, b]$?

(A) If P_1 is a finer partition than P_2, then $\|P_2\| \leq \|P_1\|$ (Here, *finer* means that, as a set, $P_2 \subseteq P_1$, i.e., all the points of P_2 are also points of P_1).
(B) If $\|P_2\| \leq \|P_1\|$, then $L_f(P_2) \leq L_f(P_1)$ (where L_f is the lower sum).
(C) If $\|P_2\| \leq \|P_1\|$, then $U_f(P_2) \leq U_f(P_1)$ (where U_f is the upper sum).
(D) If $\|P_2\| \leq \|P_1\|$, then $L_f(P_2) \leq U_f(P_1)$.
(E) All of the above.

Answer: Option (D).

Explanation: Option (D) is true for the rather trivial reason that any lower sum of f over any partition cannot be more than any upper sum of f over any partition. The norm plays no role.

Option (A) is incorrect because the inequality actually goes the other way: the finer partition has the smaller norm. Options (B) and (C) are incorrect because a smaller norm does not, in and of itself, guarantee anything about how the lower and upper sums compare.

Performance review: 4 out of 12 got this correct. 4 chose (A), 2 chose (B), 1 chose (C), 1 chose (E).

Historical note (last year): 4 out of 15 people got this correct. 8 people chose (C), presumably with the intuition that the smaller the norm of a partition, the smaller its upper sums. While this intuition is right in a broad sense, it is not correct in the precise sense that would make (C) correct. It is possible that a lot of people did not read (D) carefully, and stopped after seeing (C), which they thought was a correct statement. 1 person each chose (A), (E), and (C)+(D).

Historical note (two years ago): This question appeared in a 152 midterm two years ago, and 6 of 29 people got this right. Many people chose (C) in that test too (though I haven't preserved numerical information on number of wrong choices selected). *History repeats itself, despite my best efforts!* On the plus side, though, this is just a quiz after your first college exposure to the material, as opposed to what was a midterm last year, which happened after homeworks and a midterm review session.

Action point: Please make sure you understand very clearly the relation between the "finer" partition notion, norms of partitions, upper sums, and lower sums, to the point where you wondered how you could have ever got this question wrong.

(4) (**) Suppose F and G are continuously differentiable functions on all of $\mathbb{R}$ (i.e., both F' and G' are continuous). Which of the following is **not necessarily true**?

(A) If $F'(x) = G'(x)$ for all integers x, then $F - G$ is a constant function when restricted to integers, i.e., it takes the same value at all integers.
(B) If $F'(x) = G'(x)$ for all numbers x that are not integers, then $F - G$ is a constant function when restricted to the set of numbers x that are not integers.
(C) If $F'(x) = G'(x)$ for all rational numbers x, then $F - G$ is a constant function when restricted to the set of rational numbers.
(D) If $F'(x) = G'(x)$ for all irrational numbers x, then $F - G$ is a constant function when restricted to the set of irrational numbers.
(E) None of the above, i.e., they are all necessarily true.

Answer: Option (A).

Explanation: The fact that the derivatives of two functions agree at integers says nothing about how the derivatives behave elsewhere – they could differ quite a bit at other places. Hence, (A) is not necessarily true, and hence must be the right option. All the other options are correct as statements and hence cannot be the right option. This is because in all of them, the set of points where the derivatives agree is *dense* – it intersects every open interval. So, continuity forces the functions F' and G' to be equal everywhere, forcing $F - G$ to be constant everywhere.

Performance review: 5 out of 12 got this correct. 3 chose (D), 2 chose (C), 2 chose (E).

Historical note (last year): Nobody got this correct. 14 people chose (E) and 1 person chose (B).

Action point: This was a devilish question, because answering it correctly requires a knowledge of (or at least intuition about) ideas that you have not yet encountered. Nonetheless, it is a question whose solution you should be able to understand after having read it. Please make sure you understand why (A) is correct, i.e., how the set of integers is different from the sets in options (B), (C), and (D).

CLASS QUIZ SOLUTIONS: NOVEMBER 2: INTEGRATION

MATH 152, SECTION 55 (VIPUL NAIK)

1. Performance review

This quiz actually happened on November 7. 11 people took this quiz. The score distribution was as follows:

- Score of 0: 2 people
- Score of 1: 2 people
- Score of 2: 4 people
- Score of 3: 2 people
- Score of 4: 1 person

The mean score was 1.8. The problem wise solutions and scores:

(1) Option (B): 7 people
(2) Option (A): 7 people
(3) Option (C): 1 person
(4) Option (B): 5 people
(5) Option (E): Nobody

Overall, performance on Questions 1 and 3 was not as good as it had been last year, with Question 3 the most prominent of the lot. I think this may have been because I explained something very similar to the questions in class last year.

2. Solutions

(1) Suppose f and g are both functions on $\mathbb{R}$ with the property that f'' and g'' are both everywhere the zero function. For which of the following functions is the second derivative *not necessarily* the zero function everywhere?
   (A) $f + g$, i.e., the function $x \mapsto f(x) + g(x)$
   (B) $f \cdot g$, i.e., the function $x \mapsto f(x)g(x)$
   (C) $f \circ g$, i.e., the function $x \mapsto f(g(x))$
   (D) All of the above, i.e., the second derivative need not be identically zero for any of these functions.
   (E) None of the above, i.e., for all these functions, the second derivative is the zero function.

   Answer: Option (B)

   Explanation: f and g are both polynomial functions of degree at most 1, i.e., they are both constant or linear. A sum of two such functions is again of the same type (i.e., constant or linear). A composite of two such functions is also of the same type (i.e., constant or linear). On the other hand, a product of two such functions need not be of that type, e.g., x times x is x^2.

   Note that the question can also be solved without explicitly using the actual form of f and g, i.e., by just computing $(f + g)''$, $(f \cdot g)''$, and $(f \circ g)''$. However, this is somewhat more time-consuming.

   Performance review: 7 out of 11 got this correct. 3 chose (E) and 1 chose (D).

   Historical note (last year): 14 out of 15 people got this correct. 1 person chose (D).

(2) Suppose f and g are both functions on $\mathbb{R}$ with the property that f''' and g''' are both everywhere the zero function. For which of the following functions is the third derivative *necessarily* the zero function everywhere?
   (A) $f + g$, i.e., the function $x \mapsto f(x) + g(x)$
   (B) $f \cdot g$, i.e., the function $x \mapsto f(x)g(x)$
   (C) $f \circ g$, i.e., the function $x \mapsto f(g(x))$
   (D) All of the above, i.e., the third derivative is identically zero for all of these functions.

(E) None of the above, i.e., the third derivative is not guaranteed to be the zero function for any of these.

Answer: Option (A)

Explanation: Clearly, $(f+g)''' = f''' + g'''$, so if $f''' = g''' = 0$, then $(f+g)''' = 0$.

Counterexamples for the others: for products, take $f(x) = g(x) = x^2$. For composites, the same counterexample works. What's different from the previous problem is that while a composite of linear polynomials is linear, a composite of quadratic polynomials has degree 4.

Performance review: 7 out of 11 got this correct. 2 chose (B) and 2 chose (D).

Historical note (last year): 12 out of 15 people got this correct. 3 people chose (D).

(3) Suppose f is a function on an interval $[a, b]$, that is continuous except at finitely many interior points $c_1 < c_2 < \cdots < c_n$ ($n \geq 1$), where it has jump discontinuities (hence, both the left-hand limit and the right-hand limit exist but are not equal). Define $F(x) := \int_a^x f(t)\,dt$. Which of the following **is true**?

(A) F is continuously differentiable on (a, b) and the derivative equals f wherever f is continuous.

(B) F is differentiable on (a, b) but the derivative is not continuous, and $F' = f$ on the entire interval.

(C) F has one-sided derivatives on (a, b) and the left-hand derivative of F at any point equals the left-hand limit of f at that point, while the right-hand derivative of F at any point equals the right-hand limit of f at that point.

(D) F has one-sided derivatives on all points of (a, b) except at the points $c_1, c_2, \dots, c_n$; it is continuous at all these points but does not have one-sided derivatives.

(E) F is continuous at all points of (a, b) except at the points $c_1, c_2, \dots, c_n$.

Answer: Option (C)

Explanation: On each interval $[c_i, c_{i+1}]$, the function is differentiable on the interior and has one-sided derivatives at the endpoints, which equal the corresponding one-sided limits of f. Piecing together the intervals, we get the desired result.

Performance review: 1 out of 11 got this correct. 4 each chose (B) and (E), 2 chose (D).

Historical note (last year): 8 out of 15 people got this correct. 5 people chose (B) and 2 people chose (D).

Action point: We'll review this in class next time.

(4) (**) For a continuous function f on $\mathbb{R}$ and a real number a, define $F_{f,a}(x) = \int_a^x f(t)\,dt$. Which of the following is **true**?

(A) For every continuous function f and every real number a, $F_{f,a}$ is an antiderivative for f, and every antiderivative of f can be obtained in this way by choosing a suitably.

(B) For every continuous function f and every real number a, $F_{f,a}$ is an antiderivative for f, but it is not necessary that every antiderivative of f can be obtained in this way by choosing a suitably. (i.e., there are continuous functions f where not every antiderivative can be obtained in this way).

(C) For every continuous function f, every antiderivative of f can be written as $F_{f,a}$ for some suitable a, but there may be some choices of f and a for which $F_{f,a}$ is not an antiderivative of f.

(D) There may be some choices for f and a for which $F_{f,a}$ is not an antiderivative for f, and there may be some choices of f for which there exist antiderivatives that cannot be written in the form $F_{f,a}$.

(E) None of the above.

Answer: Option (B)

Explanation: The first clause: for every continuous function f and every real number a, $F_{f,a}$ is an antiderivative of f is just a restatement of Theorem 5.3.5, which we covered. This already whittles our options down to (A) and (B). To see why (B) is true, imagine a situation where $F_{f,0}$ does not take all real values, e.g., it is an increasing function with horizontal asymptotes at -1 and 1. We have $F_{f,0} - F_{f,a} = F_{f,0}(a)$ (by properties of integrals). Thus, there is no way we can choose a value of a for which $F_{f,a} = F_{f,0} + 5$.

In more intuitive terms, the problem is that whereas for getting all antiderivatives, we should be able to add arbitrary constants, there could be cases where the definite integral between two points cannot be made to include the set of all constants.

An explicit functional example (unfamiliar to you at this stage) is where $f(x) = 1/(x^2+1)$. Then $F_{f,0} = \arctan$ is bounded between $-\pi/2$ and $\pi/2$. Thus, say the function $20 + \arctan x$ cannot be realized as $F_{f,a}$ for any a.

Performance review: 5 out of 11 got this correct. 5 chose (C), 1 chose (A).

Historical note (last year): 5 out of 15 people got this correct. 5 people chose (A), 4 people chose (C), and 1 person chose (D).

Action point: We will return to this in Math 153 when we study improper integrals.

(5) (**) Suppose F is a differentiable function on an open interval (a, b) and F' is not a continuous function. Which of these discontinuities can F' have?

(A) A removable discontinuity (the limit exists and is finite but is not equal to the value of the function)
(B) An infinite discontinuity (one or both the one-sided limits is infinite)
(C) A jump discontinuity (both one-sided limits exist and are finite, but not equal)
(D) All of the above
(E) None of the above

Answer: Option (E)

Explanation: This is hard – perhaps part of a future challenge problem, so won't say more more. Briefly, the only kinds of discontinuities allowed are oscillatory discontinuities, of the kind seen with the derivative of $x^2 \sin(1/x)$ at 0.

Performance review: Nobody got this correct. 3 each chose (B), (C), (D), and 1 chose (A).

Historical note (last year): Nobody got this correct.

CLASS QUIZ SOLUTIONS: NOVEMBER 4: INTEGRATION

MATH 152, SECTION 55 (VIPUL NAIK)

1. Performance review

12 people took this quiz. The score distribution was as follows:

- Score of 0: 3 people
- Score of 1: 2 people
- Score of 2: 5 people
- Score of 3: 2 people

The mean score was 1.5. Here are the problem wise answers and performance:

(1) Option (A): 7 people
(2) Option (B): 4 people
(3) Option (A): 3 people
(4) Option (C): 3 people
(5) Option (C): 1 person

2. Solutions

(1) Which of the following is an **antiderivative** of $x \cos x$?
   (A) $x \sin x + \cos x$
   (B) $x \sin x - \cos x$
   (C) $-x \sin x + \cos x$
   (D) $-x \sin x - \cos x$
   (E) None of the above

   Answer: Option (A).

   Explanation: Differentiating the function given in option (A) gives $x \cos x + \sin x - \sin x = x \cos x$.

   When we study integration by parts next quarter, we will see a constructive approach designed to arrive at the answer.

   Performance review: 7 out of 11 people got this. 1 chose (B), 4 chose (E).

   Historical note (last year): 11 out of 16 people got this correct. Remaining were 2 (B) and 3 (E).

   Action point: If you got this wrong, make sure you remember and are comfortable with the differentiation rules. The time is not yet ripe to forget those.

(2) (*) Suppose F and G are two functions defined on $\mathbb{R}$ and k is a natural number such that the k^{th} derivatives of F and G exist and are equal on all of $\mathbb{R}$. Then, $F - G$ must be a polynomial function. What is the **maximum possible degree** of $F - G$? (Note: Assume constant polynomials to have degree zero)
   (A) $k - 2$
   (B) $k - 1$
   (C) k
   (D) $k + 1$
   (E) There is no bound in terms of k.

   Answer: Option (B)

   Explanation: F and G having the same k^{th} derivative is equivalent to requiring that $F - G$ have k^{th} derivative equal to zero. For $k = 1$, this gives constant functions (polynomials of degree 0). Each time we increment k, the degree of the polynomial could potentially go up by 1. Thus, the answer is $k - 1$.

Performance review: 4 out of 12 got this correct. 4 chose (E), 2 chose (C), 1 each chose (A) and (D).

Historical note (last year): 6 out of 16 people got this correct. Remaining were: 2 (A), 2 (C), 3 (D), 3 (E).

Action point: This is the kind of question you should *definitely* get right in the future. Please review the notes on repeated integration and finding functions with given k^{th} derivative. It seems like we didn't cover this well enough in class, which might be the reason for the not-so-good performance. We'll review these ideas in class Friday.

(3) (**) Suppose f is a continuous function on $\mathbb{R}$. Clearly, f has antiderivatives on $\mathbb{R}$. For all but one of the following conditions, it is possible to guarantee, without any further information about f, that there exists an antiderivative F satisfying that condition. **Identify the exceptional condition** (i.e., the condition that it may not always be possible to satisfy).
(A) $F(1) = F(0)$.
(B) $F(1) + F(0) = 0$.
(C) $F(1) + F(0) = 1$.
(D) $F(1) = 2F(0)$.
(E) $F(1)F(0) = 0$.

Answer: Option (A)

Explanation: Suppose G is an antiderivative for f. The general expression for an antiderivative is $G + C$, where C is constant. We see that for options (b), (c), and (d), it is always possible to solve the equation we obtain to get one or more real values of C. However, (a) simplifies to $G(1) + C = G(0) + C$, whereby C is canceled, and we are left with the statement $G(1) = G(0)$. If this statement is true, then *all* choices of C work, and if it is false, then *none* works. Since we cannot guarantee the truth of the statement, (a) is the exceptional condition.

Another way of thinking about this is that $F(1) - F(0) = \int_0^1 f(x)\,dx$, regardless of the choice of F. If this integral is 0, then any antiderivative works. If it is not zero, no antiderivative works.

Performance review: 3 out of 12 got this correct. 4 chose (E), 3 chose (C), 2 chose (D).

Historical note (last year): 3 out of 16 people got this correct. Remaining were: 2 (B), 3 (C), 1 (D), 7 (E).

Action point: This is the kind of question that everybody should get correct in the future. Please make sure you understand the solution process for this question.

(4) (**) Suppose $F(x) = \int_0^x \sin^2(t^2)\,dt$ and $G(x) = \int_0^x \cos^2(t^2)\,dt$. Which of the following **is true**?
(A) $F + G$ is the zero function.
(B) $F + G$ is a constant function with nonzero value.
(C) $F(x) + G(x) = x$ for all x.
(D) $F(x) + G(x) = x^2$ for all x.
(E) $F(x^2) + G(x^2) = x$ for all x.

Answer: Option (C)

Explanation: $F(x) + G(x) = \int_0^x \sin^2(t^2) + \cos^2(t^2)\,dt = \int_0^x 1\,dt = x$.

Note that it is not possible to obtain closed expressions for F and G separately, and any attempt to do so is a waste of time.

Performance review: 3 out of 12 got this correct. 5 chose (B), 2 chose (D), 1 each chose (A) and (E).

Historical note (last year): 5 out of 16 people got this correct. Remaining were: 2 (A), 5 (B), 4 (D). It is likely that many people noted that $\sin^2(t^2) + \cos^2(t^2) = 1$ but then forgot to integrate it, hence (B) as a common wrong answer.

Action point: This one shouldn't trick you again!

(5) (**) Suppose F is a function defined on $\mathbb{R} \setminus \{0\}$ such that $F'(x) = -1/x^2$ for all $x \in \mathbb{R} \setminus \{0\}$. Which of the following pieces of information is/are **sufficient** to determine F completely?
(A) The value of F at any two positive numbers.
(B) The value of F at any two negative numbers.
(C) The value of F at a positive number and a negative number.

(D) Any of the above pieces of information is sufficient, i.e., we need to know the value of F at any two numbers.

(E) None of the above pieces of information is sufficient.

Answer: Option (C)

Explanation: There are two open intervals: $(-\infty, 0)$ and $(0, \infty)$, on which we can look at F. On each of these intervals, $F(x) = 1/x+$ a constant, but the constant for $(-\infty, 0)$ may differ from the constant for $(0, \infty)$. Thus, we need the initial value information at one positive number and one negative number.

Performance review: 1 out of 12 got this correct. 10 chose (D) and 1 chose (E).

Historical note (last year): 4 out of 16 people got this correct. Remaining were: 8 (D), 4 (E). It seems that most people did not get the key idea for this question.

Action point: Once you have understood this question, you should be able to get any similar question correct in the future.

CLASS QUIZ SOLUTIONS: NOVEMBER 11: WHOPPERS

MATH 152, SECTION 55 (VIPUL NAIK)

1. Performance review

12 people took this quiz. The score distribution was as follows:

- Score of 1: 2 people
- Score of 2: 3 people
- Score of 3: 3 people
- Score of 4: 2 people
- Score of 8: 2 people

The mean score was 3.42. Here are the problem wise answers:

(1) Option (C): 3 people.
(2) Option (A): 8 people. *This was an exact replica, so good performance was expected.*
(3) Option (D): 3 people.
(4) Option (C): 4 people.
(5) Option (B): 5 people.
(6) Option (C): 4 people.
(7) Option (B): 7 people. *This was an exact replica, so good performance was expected.*
(8) Option (A): 6 people. *This was an exact replica, so good performance was expected.*
(9) Option (D): 1 person.

2. Solutions

(1) Suppose $g : \mathbb{R} \to \mathbb{R}$ is a continuous function such that $\lim_{x \to 0} g(x)/x^2 = A$ for some constant $A \neq 0$. What is $\lim_{x \to 0} g(g(x))/x^4$? *Similar to (but trickier than) October 4 Question 2: only 1 person got that right.*
(A) A
(B) A^2
(C) A^3
(D) $A^2 g(A)$
(E) $g(A)/A^2$

Answer: Option (C)

Explanation: First, note that since $g(x)/x^2 \to A$ as $x \to 0$, we must have $g(x) \to 0$ as $x \to 0$. In particular, $g(0) = 0$.

Now, consider:

$$\lim_{x \to 0} \frac{g(g(x))}{x^4} = \lim_{x \to 0} \frac{g(g(x))}{(g(x))^2} \cdot \frac{(g(x))^2}{x^4}$$

Splitting the limit, we get:

$$\lim_{x \to 0} \frac{g(g(x))}{(g(x))^2} \lim_{x \to 0} \left(\frac{g(x)}{x^2} \right)^2$$

Setting $u = g(x)$ for the first limit, and using the fact that as $x \to 0$, $u \to 0$ we see that the first limit is A. For the second limit, pulling the square out yields that the second limit is A^2. The overall limit is thus $A \cdot A^2 = A^3$.

We can also use an actual example to solve this problem. For instance, consider the extreme case where $g(x) = Ax^2$ (identically). In this case, $g(g(x)) = A(Ax^2)^2 = A^3x^4$. Thus, $g(g(x))/x^4 = A^3$, and the limit is thus A^3.

Even more generally, if $\lim_{x\to 0} g(x)/x^n = A$, then $\lim_{x\to 0} g(g(x))/x^{n^2} = A^{n+1}$.

Performance review: 3 out of 12 got this correct. 7 chose (D), 1 chose (B), 1 chose (E).

Historical note (last year): 4 out of 16 people got this correct. 5 people chose (A), 4 people chose (B), 2 people chose (D), and 1 person chose (E).

Also, this appeared in one of the error-spotting exercises for Midterm 1.

(2) Which of the following statements is **always true**? *Exact replica of a past question.*
   (A) The range of a continuous nonconstant function on a closed bounded interval (i.e., an interval of the form $[a, b]$) is a closed bounded interval (i.e., an interval of the form $[m, M]$).
   (B) The range of a continuous nonconstant function on an open bounded interval (i.e., an interval of the form (a, b)) is an open bounded interval (i.e., an interval of the form (m, M)).
   (C) The range of a continuous nonconstant function on a closed interval that may be bounded or unbounded (i.e., an interval of the form $[a, b]$, $[a, \infty)$, $(-\infty, a]$, or $(-\infty, \infty)$) is also a closed interval that may be bounded or unbounded.
   (D) The range of a continuous nonconstant function on an open interval that may be bounded or unbounded (i.e., an interval of the form (a, b),(a, ∞), $(-\infty, a)$, or $(-\infty, \infty)$), is also an open interval that may be bounded or unbounded.
   (E) None of the above.

   Answer: Option (A)

   Explanation: This is a combination of the extreme-value theorem and the intermediate-value theorem. By the extreme-value theorem, the continuous function attains a minimum value m and a maximum value M. By the intermediate-value theorem, it attains every value between m and M. Further, it can attain no other values because m is after all the minimum and M the maximum.

   The other choices:

   Option (B): Think of a function that increases first and then decreases. For instance, the function $f(x) := \sqrt{1 - x^2}$ on $(-1, 1)$ has range $(0, 1]$, which is not open. Or, the function $\sin x$ on the interval $(0, 2\pi)$ has range $[-1, 1]$.

   Option (C): We can get counterexamples for unbounded intervals. For instance, consider the function $f(x) := 1/x$ on $[1, \infty)$. The range of this function is $(0, 1]$, which is not closed. The idea is that we make the function approach but not reach a finite value as $x \to \infty$ (we'll talk more about this when we deal with asymptotes).

   Option (D): The same counterexample as for option (B) works.

   Performance review: 8 out of 12 got this correct. 1 each chose (B), (C), (D), and (E).

   Historical note (last year): 9 out of 16 people got it correct. 3 people chose (C), 2 people chose (D), and 1 person each chose (A) and (E).

   Historical note (last year, previous quiz): When the question appeared in a previous quiz, 2 out of 11 people got it correct.

(3) Suppose f is a continuously differentiable function on $\mathbb{R}$ and $c \in \mathbb{R}$. Which of the following implications is **false**? *Similar to (but trickier than) October 13 question 4.*
   (A) If f has mirror symmetry about $x = c$, f' has half turn symmetry about $(c, f'(c))$.
   (B) If f has half turn symmetry about $(c, f(c))$, f' has mirror symmetry about $x = c$.
   (C) If f' has mirror symmetry about $x = c$, f has half turn symmetry about $(c, f(c))$.
   (D) If f' has half turn symmetry about $(c, f'(c))$, f has mirror symmetry about $x = c$.
   (E) None of the above, i.e., they are all true.

   Answer: Option (D)

   Explanation: We can construct a number of examples, but instead of doing that, we make a general note. If f has mirror symmetry about $x = c$, then not only must f' have half turn symetry about $(c, f'(c))$, we must *also* have $f'(c) = 0$. Therefore, in any situation where f' has half turn symmetry about a point where it does not take the value 0, f will not have mirror symmetry about that point. More details below.

Option (A) is true. In fact, the following stronger claim is true: if f has mirror symmetry about $x = c$ and f' is defined on all of $\mathbb{R}$, then $f'(c) = 0$ and $f'(c+h) + f'(c-h) = 0$ for all $h \in \mathbb{R}$. This can be proved as follows. By mirror symmetry:

$$f(c+h) = f(c-h)$$

This is true as an identity in h. Thus, differentiating both sides with respect to h and using the chain rule, we get:

$$f'(c+h) = -f'(c-h)$$

The negative sign arises due to the chain rule.

Option (B) is true and the justification is similar to that for option (A). Namely:

$$f(c+h) + f(c-h) = 2f(c)$$

Differentiating both sides with respect to h gives:

$$f'(c+h) - f'(c-h) = 0$$

Option (C) requires more justification. The idea is that:

$$f(c+h) - f(c) - \int_c^{c+h} f'(t)\,dt$$

and:

$$f(c) - f(c-h) = \int_{c-h}^{c} f'(t)\,dt$$

Now, since f' has mirror symmetry about c, the two definite integrals are equal, so we get:

$$f(c+h) - f(c) = f(c) - f(c-h)$$

which is precisely the condition for half turn symmetry of f about $(c, f(c))$.

Option (D) is false, because, as mentioned earlier, for f to have mirror symmetry given f' having half turn symmetry, we need the additional condition that $f'(c) = 0$.

For instance, any cubic polynomial has half turn symmetry, but most polynomials of degree four do not have mirror symmetry. In fact, a polynomial of degree four has mirror symmetry iff its derivative cubic has the property that its point of inflection is a critical point.

Additional note: As we differentiate things, we obtain more and more symmetry. Here is the overall summary (where each step is true assuming that we can differentiate):

Half turn symmetry $\overset{\text{diff}}{\to}$ Mirror symmetry $\overset{\text{diff}}{\to}$ Half turn symmetry about point on x-axis.

Further, these arrows are reversible: any antiderivative of a function with half turn symmetry about a point on the x-axis has mirror symmetry about the same point, and any antiderivative of that has half turn symmetry for a point with the same x-coordinate.

The even and odd functions are special cases where the x-coordinate is 0, so in that special case:

Half turn symmetry about point on y-axis $\overset{\text{diff}}{\to}$ Even function $\overset{\text{diff}}{\to}$ Odd function

Performance review: 3 out of 12 got this correct. 5 chose (E), 3 chose (C), 1 left the question blank.

Historical note (last year): 2 out of 16 people got it correct. 7 people chose (E), 5 people chose (C), and 2 people chose (A).

(4) Consider the function $f(x) := \{ \begin{array}{rl} x, & 0 \le x \le 1/2 \\ x - (1/5), & 1/2 < x \le 1 \end{array}$. Define by $f^{[n]}$ the function obtained by iterating f n times, i.e., the function $f \circ f \circ \cdots \circ f$ where f occurs n times. What is the smallest n for which $f^{[n]} = f^{[n+1]}$? *Similar to a question on the previous midterm.*

(A) 1
(B) 2
(C) 3

(D) 4
(E) 5

Answer: Option (C)

Explanation: We need to iterate f enough times that everything gets inside $[0, 1/2]$, after which it becomes stable. Note that each time, the value goes down by 0.2. Thus, for any $x \leq 1$, we need at most three steps to bring it in $[0, 1/2]$, with the upper bound of 3 being attained for 1.

Performance review: 4 out of 12 got this correct. 3 chose (D), 2 each chose (A) and (E), 1 chose (B).

Historical note (last year): 3 out of 16 people got this correct. 8 people chose (A), 3 people chose (B), and 2 people chose (D).

Action point: It seems that most people did not figure out how composition works. Please make sure you understand at least one question like this at some point in your life.

(5) With f as in the previous question, what is the set of points in $(0, 1)$ where $f \circ f$ is not continuous?
(A) 0.5 only
(B) 0.5 and 0.7
(C) 0.5, 0.7, and 0.9
(D) 0.7 and 0.9
(E) 0.9 only

Answer: Option (B)

Explanation: The piecewise definition is:

$$(f \circ f)(x) = \{ \begin{array}{rl} x, & 0 \leq x \leq 0.5 \\ x - (1/5), & 0.5 < x \leq 0.7 \\ x - (2/5), & 0.7 < x \leq 1 \end{array}$$

We see that that points of discontinuity are 0.5 and 0.7.

Note that if we considered $f \circ f \circ f$ instead, 0.9 would also be a point of discontinuity, since, the definition to the right of 0.9 would be $x - (3/5)$.

Performance review: 5 out of 12 got this correct. 4 chose (C), 2 chose (A), 1 chose (D).

Historical note (last year): 4 out of 16 people got this correct. 9 people chose (A), 2 people chose (C), and 1 person chose (D).

Action point: Same as for previous problem – make sure you understand this clearly at least once in your life.

(6) Consider the graph of the function $f(x) := x \sin(1/(x^2 - 1))$. What can we say about the vertical and horizontal asymptotes? *This resembles a future whopper.*
(A) The graph has vertical asymptotes at $x = +1$ and $x = -1$ and horizontal asymptote (in both directions) $y = 0$.
(B) The graph has vertical asymptotes at $x = +1$ and $x = -1$ and horizontal asymptote (in both directions) $y = 1$.
(C) The graph has no vertical asymptotes and horizontal asymptote (in both directions) $y = 0$.
(D) The graph has no vertical asymptotes and horizontal asymptote (in both directions) $y = 1$.
(E) The graph has no vertical or horizontal asymptotes.

Answer: Option (C)

Explanation: The points where f is undefined are $x = \pm 1$. At both these points, the limit is undefined, but the function is bounded, because the x-part has a finite limit and the $\sin(1/(x^2 - 1))$ part is bounded in $[-1, 1]$. Thus, the function cannot have a vertical asymptote (in fact, it is oscillatory with no limit at both these points).

For the horizontal asymptote, we rewrite the limit at $+\infty$ as:

$$\lim_{x \to \infty} \frac{x}{x^2 - 1} \lim_{x \to \infty} (x^2 - 1) \sin(1/(x^2 - 1))$$

The first limit is 0. As for the second limit, putting $t = 1/(x^2 - 1)$, we see that $t \to 0^+$ as $x \to \infty$, so the second limit becomes $\lim_{t \to 0^+} (\sin t)/t = 1$. The overall limit at $+\infty$ is thus 0. A similar argument works for $-\infty$. Note that since $x^2 - 1$ has even degree, we get $\lim_{t \to 0^+} (\sin t)/t$ in this case as well.

Performance review: 4 out of 12 got this correct. 4 chose (A), 2 chose (E), 1 each chose (B) and (D).

Historical note (last year): 3 out of 16 people got this correct. 6 people chose (A), 5 people chose (B), and 1 person each chose (D) and (E).

Action point: Since the most commonly chosen incorrect answer was (A), it seems that most people figured out the horizontal asymptotes correctly. However, the vertical asymptotes confused people – not surprisingly, because they confused me too when I dug up this question from last year. Once you read the solution, however, you should be able to understand it.

(7) Suppose f and g are increasing functions from $\mathbb{R}$ to $\mathbb{R}$. Which of the following functions is *not* guaranteed to be an increasing functions from $\mathbb{R}$ to $\mathbb{R}$? *An exact replica of a past question.*
(A) $f + g$
(B) $f \cdot g$
(C) $f \circ g$
(D) All of the above, i.e., none of them is guaranteed to be increasing.
(E) None of the above, i.e., they are all guaranteed to be increasing.

Answer: Option (B)

Explanation: The problem with option (B) arises when one or both functions take negative values. For instance, consider the case $f(x) := x$ and $g(x) := x$. Both are increasing functions on all of $\mathbb{R}$. However, the pointwise product is the function $x \mapsto x^2$, which is a decreasing function for negative x.

Formally, the issue is that we cannot multiply inequalities of the form $A < B$ and $C < D$ unless we are guaranteed to be working with positive numbers.

The other choices:

Option (A): For any $x_1 < x_2$, we have $f(x_1) < f(x_2)$ and $g(x_1) < g(x_2)$. Adding up, we get $f(x_1) + g(x_1) < f(x_2) + g(x_2)$, so $(f + g)(x_1) < (f + g)(x_2)$.

Option (C): For any $x_1 < x_2$, we have $g(x_1) < g(x_2)$ since g is increasing. Now, we use the factthat f is increasing to compare its values at the two points $g(x_1)$ and $g(x_2)$, and we get $f(g(x_1)) < f(g(x_2))$. We thus get $(f \circ g)(x_1) < (f \circ g)(x_2)$.

Performance review: 7 out of 12 got this correct. 4 chose (C), 1 chose (E).

Historical note (last year): 9 out of 16 people got this correct. 3 people chose (C) and 2 people each chose (D) and (E).

Historical note (last year, previous quiz): When this question appeared earlier on October 20, only 1 out of 15 people got it correct.

(8) Suppose F and G are continuously differentiable functions on all of $\mathbb{R}$ (i.e., both F' and G' are continuous). Which of the following is **not necessarily true**? *Exact replica of a previous question.*
(A) If $F'(x) = G'(x)$ for all integers x, then $F - G$ is a constant function when restricted to integers, i.e., it takes the same value at all integers.
(B) If $F'(x) = G'(x)$ for all numbers x that are not integers, then $F - G$ is a constant function when restricted to the set of numbers x that are not integers.
(C) If $F'(x) = G'(x)$ for all rational numbers x, then $F - G$ is a constant function when restricted to the set of rational numbers.
(D) If $F'(x) = G'(x)$ for all irrational numbers x, then $F - G$ is a constant function when restricted to the set of irrational numbers.
(E) None of the above, i.e., they are all necessarily true.

Answer: Option (A).

Explanation: The fact that the derivatives of two functions agree at integers says nothing about how the derivatives behave elsewhere – they could differ quite a bit at other places. Hence, (A) is not necessarily true, and hence must be the right option. All the other options are correct as statements and hence cannot be the right option. This is because in all of them, the set of points where the derivatives agree is *dense* – it intersects every open interval. So, continuity forces the functions F' and G' to be equal everywhere, forcing $F - G$ to be constant everywhere.

Performance review: 6 out of 12 got this correct. 5 chose (E), 1 chose (D).

Historical note (last year): 10 out of 16 people got this correct. 2 people each chose (B), (D), and (E).

Historical note (last year, previous quiz): Nobody got it correct.

Action point: It's possible that many of you just remembered/revisited the question and saw that the correct answer option is (A). However, you should make sure you understand *why* the correct answer option is (A).

(9) Consider the four functions $\sin(\sin x)$, $\sin(\cos x)$, $\cos(\sin x)$, and $\cos(\cos x)$. Which of the following statements are true about their periodicity?

(A) All four functions are periodic with a period of 2π.

(B) All four functions are periodic with a period of π.

(C) $\sin(\sin x)$ and $\sin(\cos x)$ have a period of π, whereas $\cos(\sin x)$ and $\cos(\cos x)$ have a period of 2π.

(D) $\cos(\sin x)$ and $\cos(\cos x)$ have a period of π, whereas $\sin(\sin x)$ and $\sin(\cos x)$ have a period of 2π.

(E) $\sin(\sin x)$ has a period of 2π, the other three functions have a period of π.

Answer: Option (D)

Explanation: Since the inner functions in all cases have a period of 2π, it is clear that all the four functions have a period of at most 2π, in fact, the period of each divides 2π. The crucial question is which of them have the smaller period π.

Let's look at $\sin \circ \sin$ first. We have:

$$\sin(\sin(x + \pi)) = \sin(-\sin x) = -\sin(\sin x)$$

So, we see that that value at $x + \pi$ is the negative, and hence usually not the equal, of the value at x. Similarly:

$$\sin(\cos(x + \pi)) = \sin(-\cos x) = -\sin(\cos x)$$

On the other hand, for the functions that have a cos on the outside, the negative sign on the inside gets eaten up by the even nature of the outer function. For instance:

$$\cos(\sin(x + \pi)) = \cos(-\sin x) = \cos(\sin x)$$

and:

$$\cos(\cos(x + \pi)) = \cos(-\cos x) = \cos(\cos x)$$

Now, this is not a proof that π is strictly the smallest period for these functions, but that can be proved using other methods. In any case, given the choices presented, it is now easy to single out (D) as the only correct answer.

The key feature here is that both sin and cos (viewed as the inner functions of the composition) have *anti-period* π: their value gets negated after an interval of π.

The outer function cos is even, hence it converts an anti-period for the inner function into a period for the overall function. The outer function sin is odd, so it keeps anti-periods anti-periods.

Performance review: 1 out of 12 got this correct. 5 chose (A), 4 chose (B), 1 chose (C), 1 chose (E).

Historical note (last year): 5 out of 16 people got this correct. 1 person left the question blank. 7 people chose (A), 1 person chose (B), and 2 people chose (C).

Action point: This is fairly tricky to get at first sight, but you should be able to read and understand the solution.

CLASS QUIZ SOLUTIONS: NOVEMBER 16: VOLUME

MATH 152, SECTION 55 (VIPUL NAIK)

1. Performance review

11 people took this 11 question quiz. The score distribution was as follows:

- Score of 4: 1 person.
- Score of 6: 2 people.
- Score of 7: 3 people.
- Score of 8: 2 people.
- Score of 10: 3 people.

The mean score was 6.92. The problem wise answers were as follows:

(1) Option (B): 11 people
(2) Option (D): 5 people
(3) Option (B): 10 people
(4) Option (A): 8 people
(5) Option (B): 8 people.
(6) Option (C): 6 people.
(7) Option (C): 4 people.
(8) Option (C): 9 people.
(9) Option (E): 7 people.
(10) Option (A): 8 people.
(11) Option (B): 7 people.

2. Solutions

(1) Oblique cylinder:Right cylinder::
  (A) Rectangle:Square
  (B) Parallelogram:Rectangle
  (C) Disk:Circle
  (D) Triangle:Rectangle
  (E) Triangle:Square

  Answer: Option (B).

  Explanation: A right cylinder is obtained by translating a region of the plane along a direction perpendicular to the plane, while an oblique cylinder is obtained by translating along some direction, not necessarily perpendicular to the plane. Similarly, a rectangle is obtained by translating a line segment along a direction perpendicular to the line segment, while a parallelogram is obtained by translating a line segment along some direction, not necessarily perpendicular to the line segment.

  Performance review: Everybody got this correct.

  Historical note (last year): 14 out of 16 people got this correct. 2 people chose (D).

(2) Right circular cone:Right circular cylinder::
  (A) Triangle:Square
  (B) Rectangle:Square
  (C) Isosceles triangle:Equilateral triangle
  (D) Isosceles triangle:Rectangle
  (E) Isosceles triangle:Square

Answer: Option (D).

Explanation: There are two ways of seeing this. One is that if we look at a cross section containing the axis of symmetry, the cross section of a right circular cone is an isosceles triangle and the cross section of a right circular cylinder is a rectangle. Another way of thinking about this is that a rectangle is obtained by translating a line segment of fixed length in a perpendicular direction, and an isosceles triangle is obtained by translating it in a perpendicular direction while shrinking it symmetrically in a fixed proportion. Similarly, a right circular cylinder is obtained by translating a fixed disk, and a right circular cone is obtained by translating it while shrinking it.

Performance review: 5 out of 11 got this. 4 chose (A), 1 chose (E), 1 left the question blank.

Historical note (last year): 13 out of 16 people got this correct. 2 people chose (A) and 1 person chose (E).

(3) Circular disk:Circle::
(A) Hollow cylinder:Solid cylinder
(B) Solid cylinder:Hollow cylinder
(C) Cube:Cuboid (cuboid is a term for rectangular prism)
(D) Cube:Square
(E) Cube:Sphere

Answer: Option (B).

Explanation: The circular disk is the region enclosed by the circle. Similarly, the solid cylinder is the region enclosed by the hollow cylinder.

Performance review: 10 out of 11 people got this. 1 chose (D).

Historical note (last year): 8 out of 16 people got this correct. 4 people chose (A) (the inverted option), 3 people chose (D), and 1 person chose (E).

(4) Circular disk:Line segment::
(A) Solid sphere:Circular disk
(B) Circle:Rectangle
(C) Sphere:Cube
(D) Cube:Right circular cylinder
(E) Square:Triangle

Answer: Option (A).

Explanation: There are many many ways of seeing this, none of which is obvious.

In terms of cross sections, a line segment is a one-dimensional cross section (i.e., intersection with a line) of a circular disk, while a circular disk is a two-dimensional cross section (i.e., intersection with a plane) of a solid sphere. Thus, a circular disk is to a solid sphere what a line segment is to a circular disk.

Better, a line segment can be described as the set of points on a line (one-dimensional space) of distance at most a certain length from a fixed point. A circular disk is the analogue in two dimensions, and a solid sphere is the analogue in three dimensions. Thus, a solid sphere is to a circular disk what a circular disk is to a line segment.

Performance review: 8 out of 11 got this. 2 chose (A), 1 chose (D).

Historical note (last year): 14 out of 16 got this correct. 1 person chose (D) and 1 person left the question blank. The number of correct answers was surprisingly large given that the logic of the analogy is far from obvious, but many people probably used the difference of dimensions.

(5) Suppose a filled triangle ABC in the plane is revolved about the side AB. Which of the following best describes the solid of revolution thus obtained if both the angles A and B are acute (ignoring issues of boundary inclusion/exclusion)?
(A) It is a right circular cone.
(B) It is the union of two right circular cones sharing a common disk as base.
(C) It is the set difference of two right circular cones sharing a common disk as base.
(D) It is the union of two right circular cones sharing a common vertex.
(E) It is the set difference of two right circular cones sharing a common vertex.

Answer: Option (B)

Explanation: Let D be the foot of the perpendicular from C to AB. Since both the angles A and B are cute, D lies on the line segment AB. Then, the triangle ABC is the union of the right triangles ACD and BCD, sharing a common side CD. Each of these right triangles gives as its solid of revolution a right circular cone, with the base disk being the disk corresponding to CD in both. Thus, the overall solid is the union of the two right circular cones with a common disk.

Performance review: 8 out of 11 got this. 2 chose (A), 1 chose (E).

Historical note (last year): 13 out of 16 people got this correct. 2 people chose (D) and 1 person chose (A).

(6) Suppose a filled triangle ABC in the plane is revolved about the side AB. Which of the following best describes the solid of revolution thus obtained if the angle A is obtuse (ignoring issues of boundary inclusion/exclusion)?
(A) It is a right circular cone.
(B) It is the union of two right circular cones sharing a common disk as base.
(C) It is the set difference of two right circular cones sharing a common disk as base.
(D) It is the union of two right circular cones sharing a common vertex.
(E) It is the set difference of two right circular cones sharing a common vertex.

Answer: Option (C) (this is not quite precise language, because we need to be careful about boundaries, but it is basically correct).

Explanation: Let D be the foot of the perpendicular from C to the line AB. Unlike the previous case, D does *not* lie on the line segment AB, because the angle A is obtuse. In fact, it lies on the A-side of the line segment. Thus, the triangle ABC is the set difference of the right triangles BCD and ACD, sharing a common side CD (modulo some boundaries getting re-included). The corresponding solid of revolution is the set difference of the corresponding right circular cones, both of which have a common base disk corresponding to the side CD.

Performance review: 6 out of 11 got this. 3 chose (E), 1 chose (B), 1 chose (D).

Historical note (last year): 9 out of 16 people got this correct. 5 people chose (D) and 1 person each chose (B) and (E).

(7) What is the volume of the solid of revolution obtained by revolving the filled triangle ABC about the side AB, if the length of the base AB is b and the height corresponding to this base is h?
(A) $(1/6)\pi b^{3/2}h^{3/2}$
(B) $(1/3)\pi b^2 h$
(C) $(1/3)\pi b h^2$
(D) $(2/3)\pi b^2 h$
(E) $(2/3)\pi b h^2$

Answer: Option (C).

Explanation: The region is a union or difference of two right circular cones, as seen in some earlier multiple choice questions. Both these cones have *radius* h. The sum or difference of the heights of these cones is b. Thus, the formula gives (C).

For the next two questions, suppose Ω is a region in a plane Π and ℓ is a line on Π such that Ω lies completely on one side of ℓ (in particular, it does not intersect ℓ). Let Γ be the solid of revolution obtained by revolving Ω about ℓ. Suppose further that the intersection of Ω with any line perpendicular to ℓ is either empty or a point or a line segment.

Performance review: 4 out of 11 got this. 3 chose (C), 3 chose (E), 1 left the question blank.

Historical note: 10 out of 16 people got this correct. 2 people each chose (B) and (D) and 1 person each chose (A) and (E).

(8) (*) What is the intersection of Γ with Π (your answer should be always true)?
(A) It is precisely Ω.
(B) It is the union of Ω and a translate of Ω along a direction perpendicular to ℓ.
(C) It is the union of Ω and the reflection of Ω about ℓ.
(D) It is either empty or a rectangle whose dimensions depend on Ω.
(E) It is either empty or a circle or an annulus whose inner and outer radius depend on Ω.

Answer: Option (C).

Explanation: The intersection of Γ with π comprises those regions obtained by revolving Ω that land inside π. This is precisely Ω and the region obtained by revolving Ω by the *angle* π (180 degrees; unfortunately there's symbol overloading here), which is equivalent to the reflection of Ω about ℓ.

Performance review: 9 out of 11 got this correct. 2 chose (E).

Historical note (last year): 6 out of 16 people got this correct. 4 people chose (A), 3 people chose (E), 2 people chose (D), and 1 person chose (B).

(9) What is the intersection of Γ with a plane perpendicular to ℓ (your answer should be always true)?
(A) It is precisely Ω.
(B) It is the union of Ω and a translate of Ω along a direction perpendicular to ℓ.
(C) It is the union of Ω and the reflection of Ω about ℓ.
(D) It is either empty or a rectangle whose dimensions depend on Ω.
(E) It is either empty or a circle or an annulus whose inner and outer radius depend on Ω.

Answer: Option (E) (Oops, *circle* should have been *circular disk*)

Explanation: This is precisely the annulus procedure used to justify the washer method.

Note that the description given by option (D) requires Ω to satisfy the condition given in the correction, which essentially says that the slices perpendicular to ℓ are nice enough.

Performance review: 7 out of 11 go this correct. 2 chose (B) and 2 chose (C).

Historical note: 9 out of 16 people got this correct. 3 people chose (C), 2 people chose (B), 1 person chose (D), and 1 person left the question blank.

(10) (*) Consider a fixed equilateral triangle ABC. Now consider, for any point D outside the plane of ABC, the solid tetrahedron $ABCD$. This is the solid bounded by the triangles ABC, BCD, ACD, and ABD. The volume of this solid depends on D. What specific information about D completely determines the volume?
(A) The perpendicular distance from D to the plane of the triangle ABC.
(B) The minimum of the distances from D to points in the filled triangle ABC.
(C) The location of the point E in the plane of triangle ABC that is the foot of the perpendicular from D to ABC.
(D) The distance from D to the center of ABC (here, you can take the center as any of the notions of center since ABC is equilateral).
(E) None of the above.

Answer: Option (A).

Explanation: In fact, the volume is 1/3 times the area of the base (which is fixed) times this perpendicular distance.

Performance review: 8 out of 11 got this correct. 2 chose (D) and 1 chose (E).

Historical note (last year): 7 out of 16 people got this correct. 3 people chose (C), 3 people chose (D), and 1 person each chose (B) and (E).

(11) (**) For $r > 0$, consider the region $\Omega_r(a)$ bounded by the x-axis, the curve $y = x^{-r}$, and the lines $x = 1$ and $x = a$ with $a > 1$. Let $V_r(a)$ be the volume of the region obtained by revolving $\Omega_r(a)$ about the x-axis. What is the precise set of values of r for which $\lim_{a\to\infty} V_r(a)$ is finite?
(A) All $r > 0$
(B) $r > 1/2$
(C) $r > 1$
(D) $r > 2$
(E) No value of r

Answer: Option (B).

Explanation: $V_r(a) = \pi \int_1^a x^{-2r}\,dx$. The limit is finite iff $2r > 1$, which is equivalent to $r > 1/2$.

Performance review: 7 out of 11 got this correct. 3 chose (A) and 1 chose (C).

Historical note (last year): 3 out of 16 people got this correct. 7 people chose (C), 4 people chose (A), and 2 people chose (E).

CLASS QUIZ SOLUTIONS: NOVEMBER 21: ONE-ONE FUNCTIONS

MATH 152, SECTION 55 (VIPUL NAIK)

1. Performance review

12 people attempted this quiz. The score distribution was as follows:

- Score of 1: 3 people.
- Score of 2: 6 people.
- Score of 3: 3 people.

The mean score was 2.
Here are the problem wise solutions and scores:

(1) Option (B): 11 people
(2) Option (C): 2 people
(3) Option (E): 9 people
(4) Option (C): 0 people. *Whoops!*
(5) Option (E): 2 people

2. Solutions

(1) For one of these function types for a continuous function from $\mathbb{R}$ to $\mathbb{R}$, it is *possible* to also be a one-to-one function. What is that function type?
  (A) Function whose graph has mirror symmetry about a vertical line.
  (B) Function whose graph has half turn symmetry about a point on it.
  (C) Periodic function.
  (D) Function having a point of local minimum.
  (E) Function having a point of local maximum.

  Answer: Option (B)

  Explanation: Think $f(x) = x$, $f(x) = x^3$, or $f(x) = x - \sin x$.

  For all the others, the violation of one-to-one is clear: periodic functions repeat completely after an interval, functions whose graph has mirror symmetry have the same value at equal distances from the axis of mirror symmetry, and the existence of a local extremum implies that values very close to that are attained both on the immediate left and the immediate right of the extremum.

  Performance review: 11 out of 12 got this correct. 1 left the question blank.

  Historical note (last year): Everybody got this correct!

(2) (**) Suppose f, g, and h are continuous one-to-one functions whose domain and range are both $\mathbb{R}$. **What can we say** about the functions $f + g$, $f + h$, and $g + h$?
  (A) They are all continuous one-to-one functions with domain $\mathbb{R}$ and range $\mathbb{R}$.
  (B) At least two of them are continuous one-to-one functions with domain $\mathbb{R}$ and range $\mathbb{R}$ – however, we cannot say more.
  (C) At least one of them is a continuous one-to-one function with domain $\mathbb{R}$ and range $\mathbb{R}$ – however, we cannot say more.
  (D) Either all three sums are continuous one-to-one functions whose domain and range are both $\mathbb{R}$, or none is.
  (E) It is possible that none of the sums is a continuous one-to-one function whose domain and range are both $\mathbb{R}$; it is also possible that one, two, or all the sums are continuous one-to-one functions whose domain and range are both $\mathbb{R}$.

Answer: Option (C)

Explanation: Since f, g, and h are all continuous one-to-one functions with domain and range $\mathbb{R}$, each one of them is either increasing or decreasing. We consider various cases:

- If all three functions are increasing, so are all the pairwise sums, and hence, all the sums $f + g$, $f + h$, and $g + h$ are increasing. Further, the domain and range of all three pairwise sums is $\mathbb{R}$.
- If all three functions are decreasing, so are all the pairwise sums, and hence, all the sums $f + g$, $f + h$, and $g + h$ are decreasing. Further, the domain and range of all three pairwise sums is $\mathbb{R}$
- If two of the functions are increasing and the third function is decreasing, then we know for certain that the sum of the two increasing functions is increasing and hence one-to-one. But the sum of either of the increasing functions with the decreasing function may be increasing, decreasing, or neither. For instance, if $f(x) = g(x) = x$ and $h(x) = -x$, then $f+h$ and $g+h$ are both the zero function, which is neither increasing nor decreasing, and hence not one-to-one.
- If two of the functions are decreasing and the third function is increasing, then we know for certain that the sum of the two dcereasing functions is decreasing and hence one-to-one. We cannot say anything for sure about the other two sums, for the same reasons as in the previous case.

It's clear from all these that (C) is the right option.

Performance review: 2 out of 12 got this. 6 chose (E), 2 chose (A), 1 each chose (B) and (D).

Historical note (last year): 2 out of 15 people got this correct. 8 people chose (A), 1 person chose (B), and 4 people chose (E).

Action point: Please make sure you understand this solution really well! This kind of question should not trip you in the future.

(3) (**) Suppose f is a one-to-one function with domain a closed interval $[a, b]$ and range a closed interval $[c, d]$. Suppose t is a point in (a, b) such that f has left hand derivative l and right-hand derivative r at t, with both l and r nonzero. What is the left hand derivative and right hand derivative to f^{-1} at $f(t)$?

(A) The left hand derivative is $1/l$ and the right hand derivative is $1/r$.
(B) The left hand derivative is $-1/l$ and the right hand derivative is $-1/r$.
(C) The left hand derivative is $1/r$ and the right hand derivative is $1/l$.
(D) The left hand derivative is $-1/r$ and the right hand derivative is $-1/l$.
(E) The left hand derivative is $1/l$ and the right hand derivative is $1/r$ if $l > 0$, otherwise the left hand derivative is $1/r$ and the right hand derivative is $1/l$.

Answer: Option (E)

Explanation: Although it isn't necessary to note this, a one-to-one function that satisfies the intermediate value property is continuous, so even though f is not explicitly given to be continuous, it is in fact continuous on its domain.

If $l > 0$, then, since we are dealing with a one-to-one function, the function is increasing throughout, and so $r \geq 0$ as well. Since we know $r \neq 0$, we conclude that $r > 0$ strictly. The upshot is that as $x \to t^-$, $f(x) \to f(t)^-$ and as $x \to t^+$, $f(x) \to f(t)^+$. Thus, when we pass to the inverse function, the roles of left and right remain the same.

On the other hand, if $l < 0$, then as $x \to t^-$, $f(x) \to f(t)^+$, and hence the roles of left and right get interchanged.

Performance review: 9 out of 12 got this. 2 chose (C) and 1 chose (A).

Historical note (last year): 6 out of 15 people got this correct. 6 people chose (A) and 3 people chose (C).

Action point: One-sided derivatives and increasing/decreasing functions are a potent mix. We've tried and failed to understand this mix many times in the past. But this might well be the time it finally clicks! Here's a repetition: when we apply an increasing function, *left remains left* and *right remains right*. But when we apply a decreasing function, *left becomes right* and *right becomes left*. Keep chanting this again and again until you understand, appreciate, and *believe* it.

(4) (**) Which of these functions is one-to-one?

(A) $f_1(x) := \begin{cases} x, & x \text{ rational} \\ x^2, & x \text{ irrational} \end{cases}$

(B) $f_2(x) := \begin{cases} x, & x \text{ rational} \\ x^3, & x \text{ irrational} \end{cases}$

(C) $f_3(x) := \begin{cases} x, & x \text{ rational} \\ 1/(x-1), & x \text{ irrational} \end{cases}$

(D) All of the above

(E) None of the above

Answer: Option (C)

Explanation: Option (A) is easy to rule out: $\sqrt{2}$ and $-\sqrt{2}$ map to the same thing. Option (B) is a little harder to rule out, because the function is one-to-one within each piece, i.e., no two rationals map to the same thing and no two irrationals map to the same thing. However, a rational and an irrational can map to the same thing. For instance, 2 and $2^{1/3}$ both map to 2.

For option (C), note that not only is the map one-to-one in each piece, but also, the image of the rationals stays inside the rationals and the image of the irrationals stays inside the irrationals. In particular, this means that a rational number and an irrational number cannot map to the same thing, so the function is globally one-to-one.

Performance review: Nobody got this correct! 10 chose (C), 2 chose (A).

Historical note (last year): 2 out of 15 people got this correct. 7 people chose (E), 4 people chose (B), and 2 people chose (D). It is possible that some of those who chose (E) lost hope after looking at (A) and (B) and concluded that checking (C) is pointless too.

Action point: This one should not trip you in the future either!

(5) (**) Consider the following function $f : [0,1] \to [0,1]$ given by $f(x) := \begin{cases} \sin(\pi x/2), & 0 \le x \le 1/2 \\ \sqrt{x}, & 1/2 < x \le 1 \end{cases}$.

What is the correct expression for $(f^{-1})'(1/2)$?

(A) It does not exist, since the two-sided derivatives of f at $1/2$ do not match.

(B) $\sqrt{2}$

(C) $2\sqrt{2}/\pi$

(D) $4/\pi$

(E) $4/(\sqrt{3}\pi)$

Answer: Option (E)

Explanation: We use:

$$(f^{-1})'(1/2) = \frac{1}{f'(f^{-1}(1/2))}$$

By inspection, we see that $f^{-1}(1/2)$ must be between 0 and $1/2$. Thus, we must solve $\sin(\pi x/2) = 1/2$. This gives $\pi x/2 = \pi/6$ (considering domain restrictions) so $x = 1/3$. Thus, we get:

$$(f^{-1})'(1/2) = \frac{1}{f'(1/3)}$$

The expression for the derivative is $(\pi/2)\cos(\pi x/2)$, which evaluated at $1/3$ gives $(\pi\sqrt{3})/4$. Taking the reciprocal, we get $4/(\pi\sqrt{3})$.

Note that (A) is a sophisticated distractor in the sense that if you naively consider:

$$(f^{-1})'(1/2) = \frac{1}{f'(1/2)}$$

You will wrongly conclude (A). (B) and (C) are the one-sided derivative at $f(1/2)$, so these too are attractive propositions for the naive.

Performance review: 2 out of 12 got this. 4 chose (A), 3 chose (D), 2 chose (C), 1 chose (B).

Historical note (last year): 1 out of 15 people got this correct. 7 people chose (A), 5 people chose (C), and 2 people chose (B).

Action point: I emphasized this point in class: $(f^{-1})'(x) = 1/(f'(f^{-1}(x)))$, not $1/(f'(x))$. However, the sting of getting it wrong on a quiz might be a greater spur to remember this fact forever. Make sure you never fall for this error again!

CLASS QUIZ SOLUTIONS: NOVEMBER 23: MEMORY LANE

MATH 152, SECTION 55 (VIPUL NAIK)

1. Performance review

12 people took this quiz. The score distribution was as follows:

- Score of 2: 5 people
- Score of 3: 3 people
- Score of 4: 2 people
- Score of 6: 1 person
- Score of 7: 1 person

The mean score was 3.33. The problem wise answers and performance are as follows:

(1) Option (B): 5 people
(2) Option (E): 7 people
(3) Option (D): 9 people
(4) Option (C): 7 people
(5) Option (D): 5 people
(6) Option (D): 2 people
(7) Option (A): 5 people

2. Solutions

(1) For which of the following specifications is there **no continuous function** satisfying the specifications?
   (A) Domain $[0, 1]$ and range $[0, 1]$
   (B) Domain $[0, 1]$ and range $(0, 1)$
   (C) Domain $(0, 1)$ and range $[0, 1]$
   (D) Domain $(0, 1)$ and range $(0, 1)$
   (E) None of the above, i.e., we can get a continuous function for each of the specifications.

   Answer: Option (B)

   Explanation: By the extreme value theorem, any continuous function on a closed bounded interval must attain its maximum and minimum, and hence its image cannot be an open interval.

   The other choices:

   For options (A) and (D), we can pick the identity functions $f(x) := x$ on the respective domains.

   For option (C), we can pick the function $f(x) := \sin^2(2\pi x)$ on the domain $(0, 1)$.

   Performance review: 5 out of 12 got this. 6 chose (C), 1 chose (D).

   Historical note (last year): 7 out of 14 people got this correct. 5 people chose (C) and 2 people chose (E).

   Action point: Any question that involves feasible options for the range of a function should remind you of the *intermediate value theorem* and *extreme value theorem*. It seems likely that the people who got this question wrong (and perhaps some of the point who got it right too!) did not even think of the extreme value theorem.

(2) Suppose f and g are continuous functions on $\mathbb{R}$, such that f is continuously differentiable everywhere and g is continuously differentiable everywhere except at c, where it has a vertical tangent. What can we say is **definitely true** about $f \circ g$?
   (A) It has a vertical tangent at c.
   (B) It has a vertical cusp at c.
   (C) It has either a vertical tangent or a vertical cusp at c.

(D) It has neither a vertical tangent nor a vertical cusp at c.
(E) We cannot say anything for certain.

Answer: Option (E).

Explanation: Consider $g(x) := x^{1/3}$. This has a vertical tangent at $c = 0$. If we choose $f(x) = x$, we get (A). If we choose $f(x) = x^2$, we get (B). If we choose $f(x) = x^3$, we get neither a vertical tangent nor a vertical cusp. Hence, (E) is the only viable option.

Performance review: 7 out of 12 got this correct. 4 chose (C), 1 chose (D).

Historical note (last year): 5 out of 14 people got this correct. Other choices were (A) (3), (C) (4), (B) (1),and (D) (1).

Historical note: In an earlier quiz where this question appeared, 3 out of 15 people got this correct. Other choices were (A) (7), (C) (4), and (D) (1). The main thing that people had trouble with was thinking of possibilities for f that could play the role of converting the vertical tangent behavior of the original function g into vertical cusp or "neither" behavior for the composite function.

Action point: Performance this time was a little better than earlier, but it seems that many of you either did not read the original solution or it did not register properly in your minds. Well, there's always a second chance! Take it this time.

(3) Consider the function $p(x) := x^{2/3}(x-1)^{3/5} + (x-2)^{7/3}(x-5)^{4/3}(x-6)^{4/5}$. For what values of x does the graph of p have a vertical cusp at $(x, p(x))$?
(A) $x = 0$ only.
(B) $x = 0$ and $x = 5$ only.
(C) $x = 5$ and $x = 6$ only.
(D) $x = 0$ and $x = 6$ only.
(E) $x = 0$, $x = 5$, and $x = 6$.

Answer: Option (D)

Explanation: This uses local behavior heuristics, both additive and multiplicative. We need the exponent on top to be p/q where $0 < p < q$ with p even and q odd.

Performance review: 9 out of 12 got this correct. 1 each chose (A), (C), and (E).

Historical note (last year): 3 out of 14 people got this correct. 5 people chose (E) (indcating that they probably forgot the condition that $p < q$), 4 people chose (C), 3 people chose (B), and 1 person chose (A).

Action point: Review the local behavior heuristics section of the review sheet for midterm 2. Or, if this was just a careless error about not noting that a particular number was bigger than 1, don't make the careless error again.

(4) Consider the function $f(x) := \{ \begin{array}{ll} x, & 0 \le x \le 1/2 \\ x^2, & 1/2 < x \le 1 \end{array}$. What is $f \circ f$?

(A) $x \mapsto \{ \begin{array}{ll} x, & 0 \le x \le 1/2 \\ x^4, & 1/2 < x \le 1 \end{array}$

(B) $x \mapsto \{ \begin{array}{ll} x, & 0 \le x \le 1/2 \\ x^2, & 1/2 < x \le 1 \end{array}$

(C) $x \mapsto \{ \begin{array}{ll} x, & 0 \le x \le 1/2 \\ x^2, & 1/2 < x \le 1/\sqrt{2} \\ x^4, & 1/\sqrt{2} < x \le 1 \end{array}$

(D) $x \mapsto \{ \begin{array}{ll} x, & 0 \le x \le 1/\sqrt{2} \\ x^2, & 1/\sqrt{2} < x \le 1 \end{array}$

(E) $x \mapsto \{ \begin{array}{ll} x, & 0 \le x \le 1/\sqrt{2} \\ x^4, & 1/\sqrt{2} < x \le 1 \end{array}$

Answer: Option (C)

Explanation: If $0 \le x \le 1/2$, then $f(x) = x$, so $f(f(x)) = x$. If $1/2 < x \le 1$, then $f(x) = x^2$. What happens when we apply f to that depends on where x^2 falls. If $0 \le x^2 \le 1/2$, then $f(x^2) = x^2$, so $f(f(x)) = x^2$. This covers $1/2 < x \le 1/\sqrt{2}$. Otherwise $f(x^2) = x^4$, so $f(f(x)) = x^4$.

Performance review: 7 out of 12 got this correct. 3 chose (A), 1 chose (A), 1 chose (B).

Historical note (last year): 4 out of 14 people got this correct. 4 people chose (E), 4 people chose (A), 1 person chose (D), and 1 person left the question blank.

Action point: It seems that many people don't have the correct conceptual picture of how to compose functions with piecewise definitions. *You need to spend some time to understand this – please do!* We will talk briefly about this in one of the subsequent review opportunities.

(5) Suppose f and g are functions $(0,1)$ to $(0,1)$ that are both right continuous on $(0,1)$. Which of the following is *not* guaranteed to be right continuous on $(0,1)$?
(A) $f+g$, i.e., the function $x \mapsto f(x)+g(x)$
(B) $f-g$, i.e., the function $x \mapsto f(x)-g(x)$
(C) $f \cdot g$, i.e., the function $x \mapsto f(x)g(x)$
(D) $f \circ g$, i.e., the function $x \mapsto f(g(x))$
(E) None of the above, i.e., they are all guaranteed to be right continuous functions

Answer: Option (D)

Explanation: See the explanation for Question 2 on the October 1 quiz. Note that that quiz uses left continuity, but the example can be adapted to right continuity.

Performance review: 5 out of 12 got this correct. 4 chose (E), 3 chose (C).

Historical note (last year): 9 out of 14 people got the question correct. 3 people chose (E) and 1 person each chose (B) and (C).

(6) For a partition $P = x_0 < x_1 < x_2 < \cdots < x_n$ of $[a,b]$ (with $x_0 = a$, $x_n = b$) define the norm $\|P\|$ as the maximum of the values $x_i - x_{i-1}$. Which of the following **is always true** for any continuous function f on $[a,b]$? (5 points)
(A) If P_1 is a finer partition than P_2, then $\|P_2\| \le \|P_1\|$ (Here, *finer* means that, as a set, $P_2 \subseteq P_1$, i.e., all the points of P_2 are also points of P_1).
(B) If $\|P_2\| \le \|P_1\|$, then $L_f(P_2) \le L_f(P_1)$ (where L_f is the lower sum).
(C) If $\|P_2\| \le \|P_1\|$, then $U_f(P_2) \le U_f(P_1)$ (where U_f is the upper sum).
(D) If $\|P_2\| \le \|P_1\|$, then $L_f(P_2) \le U_f(P_1)$.
(E) All of the above.

Answer: Option (D).

Explanation: Option (D) is true for the rather trivial reason that any lower sum of f over any partition cannot be more than any upper sum of f over any partition. The norm plays no role.

Option (A) is incorrect because the inequality actually goes the other way: the finer partition has the smaller norm. Options (B) and (C) are incorrect because a smaller norm does not, in and of itself, guarantee anything about how the lower and upper sums compare.

Performance review: 2 out of 12 got this correct. 5 chose (C), 3 chose (B), 1 each chose (A) and (E).

Historical note (last year): 9 out of 14 people got this correct. 3 people chose (B) and 1 person each chose (A) and (C).

Historical note 1: In the previous quiz appearance, 4 out of 15 prople got this correct. 8 people chose (C), presumably with the intuition that the smaller the norm of a partition, the smaller its upper sums. While this intuition is right in a broad sense, it is not correct in the precise sense that would make (C) correct. It is possible that a lot of people did not read (D) carefully, and stopped after seeing (C), which they thought was a correct statement. 1 person each chose (A), (E), and (C)+(D).

Historical note 2: This question appeared in a 152 midterm two years ago, and 6 of 29 people got this right. Many people chose (C) in that test too (though I haven't preserved numerical information on number of wrong choices selected).

(7) A disk of radius r in the xy-plane is translated parallel to itself with its center moving in the yz-plane along the semicircle $y^2 + z^2 = R^2, y \ge 0$. The solid thus obtained can be thought of as a *cylinder of bent spine* with cross sections being disks of radius r along the xy-plane and the centers forming a semicircle of radius R in the yz-plane, with the z-value ranging from $-R$ to R. What is the volume of this solid?
(A) $2\pi r^2 R$
(B) $\pi^2 r^2 R$

(C) $2\pi r R^2$
(D) $\pi^2 r R^2$
(E) $\pi^2 R^3$

Answer: Option (A)

Explanation: For $-R \leq z \leq R$, the cross section in the xy-plane has constant area with value πr^2. Thus, the total volume is $\pi r^2 \times (R - (-R)) = \pi r^2 \times 2R = 2\pi r^2 R$.

Performance review: 5 out of 12 got this correct. 3 chose (C), 2 chose (B), 1 each chose (D) and (E).

Historical note (last year): 3 out of 14 people got this correct. 5 people chose (C), 4 people chose (B), and 2 people chose (E).

Action point: Make sure you understand this really really well! In particular, make sure you understand why this is *not*, repeat *not*, a solid of revolution but rather a *cylinder with bent spine*.

CLASS QUIZ SOLUTIONS: NOVEMBER 28: LOGARITHM AND EXPONENTIAL

MATH 152, SECTION 55 (VIPUL NAIK)

1. Performance review

11 people took this 6-question quiz. The score distribution was as follows:

- Score of 2: 1 person
- Score of 4: 6 people
- Score of 5: 4 people

The mean score was 3.83. The answers and performance review are as follows:

(1) Option (B): 3 people
(2) Option (A): 9 people
(3) Option (A): 10 people
(4) Option (C): 2 people
(5) Option (B): 11 people
(6) Option (B): 11 people

2. Solutions

(1) Consider the function $f(x) := \exp(5 \ln x)$ defined for $x \in (0, \infty)$. How does $f(x)$ grow as a function of x?
   (A) As a linear function
   (B) As a polynomial function but faster than a linear function
   (C) Faster than a polynomial function but slower than an exponential function
   (D) As an exponential function, i.e., $x \mapsto \exp(kx)$ for some $k > 0$
   (E) Faster than an exponential function

   Answer: Option (B)

   Explanation: $\exp(5 \ln x) = (\exp(\ln x))^5 = x^5$ is a polynomial in x.

   Performance review: 3 people got this correct. 3 chose (C), 3 chose (D), 1 chose (E).

(2) Consider the function $f(x) := \ln(5 \exp x)$ for $x \in (0, \infty)$. How does $f(x)$ grow as a function of x?
   (A) As a linear function
   (B) As a polynomial function but faster than a linear function
   (C) Faster than a polynomial function but slower than an exponential function
   (D) As an exponential function, i.e., $x \mapsto \exp(kx)$ for some $k > 0$
   (E) Faster than an exponential function

   Answer: Option (A)

   Explanation: $\ln(5 \exp x) = \ln 5 + \ln(\exp x) = \ln 5 + x$, which is a linear function.

   Performance review: 9 people got this correct. 2 people chose (B).

(3) Consider the function $f(x) := \ln((\exp x)^5)$ defined for $x \in (0, \infty)$. How does $f(x)$ grow as a function of x?
   (A) As a linear function
   (B) As a polynomial function but faster than a linear function
   (C) Faster than a polynomial function but slower than an exponential function
   (D) As an exponential function, i.e., $x \mapsto \exp(kx)$ for some $k > 0$
   (E) Faster than an exponential function

   Answer: Option (A)

   Explanation: $\ln((\exp x)^5) = \ln(\exp(5x)) = 5x$ is a linear function of x.

   Performance review: 10 people got this correct. 1 chose (C).

(4) Consider the function $f(x) := \exp((\ln x)^5)$ defined for $x \in (0, \infty)$. How does $f(x)$ grow as a function of x?
(A) As a linear function
(B) As a polynomial function but faster than a linear function
(C) Faster than a polynomial function but slower than an exponential function
(D) As an exponential function, i.e., $x \mapsto \exp(kx)$ for some $k > 0$
(E) Faster than an exponential function
Answer: Option (C)
Explanation: This is a little tricky, so we break it down into two parts.
First, note that any polynomial function (with positive leading coefficient) grows like x^n for n a positive integer. The log of that grows like $\ln(x^n) = n \ln x$, i.e., as a linear function of $\ln x$. This is slower than $(\ln x)^5$. Thus, the polynomial function grows slower than $\exp((\ln x)^5)$.
Second, note that an exponential function in x is something like $\exp x$ or $\exp(kx)$, and x or kx grows faster than any power of $\ln x$, so the function $\exp x$ grows faster than $\exp((\ln x)^5)$.
Performance review: 2 people got this correct. 6 chose (E), 2 chose (D), 1 chose (B).

(5) *Consumption smoothing*: A certain measure of happiness is found to be a logarithmic function of consumption, i.e., the happiness level H of a person is found to be of the form $H = a + b \ln C$ where C is the person's current consumption level, and a and b are positive constants independent of the consumption level.
The person has a certain total consumption C_{tot} to be split within two years, year 1 and year 2, i.e., $C_{tot} = C_1 + C_2$. Thus, the person's happiness level in year 1 is $H_1 = a + b \ln C_1$ and the person's happiness level in year 2 is $H_2 = a + b \ln C_2$. How would the person choose to split consumption between the two years to maximize average happiness across the years?
(A) All the consumption in either one year
(B) Equal amount of consumption in the two years
(C) Consume twice as much in one year as in the other year
(D) Consumption in the two years is in the ratio $a : b$
(E) It does not matter because any choice of split of consumption level between the two years produces the same average happiness
Answer: Option (B)
Explanation: Basically, happiness is logarithmic in consumption, so if consumption is unequal, then it can be distributed from the higher consumption year to the lower consumption year. The *fractional* loss in the higher consumption year is lower than the *fractional* gain in the lower consumption year. The nature of logarithmis means that the *absolute* loss in the higher consumption year is lower than the *absolute* gain in the lower consumption year. The process continues till consumption in both years is exactly equal.
We can also do this formally. We are basically using the fact that the logarithm function is concave down.
Performance review: Everybody got this correct.

(6) *Income inequality and subjective well being*: Subjective well being *across* individuals is found to be logarithmically related to income. Every doubling of income is found to increase an individuals' measured subjective well being by 0.3 points on a certain scale. *Holding total income across two individuals constant*, how should that income be divided between the two individuals to maximize their average subjective well being?
(A) All the income goes to one person
(B) Both earn the exact same income
(C) One person earns twice as much as the other
(D) One person earns 0.3 times as much as the other
(E) It does not matter because the average subjective well being is independent of the distribution of income.
Answer: Option (B)

Explanation: The logic is *exactly* the same as the preceding question, except that instead of an individual distributing consumption between years, income is being "distributed" between individuals in the same year, and instead of happiness, we are measuring subjective well being.

Performance review: Everybody got this correct.

CLASS QUIZ SOLUTIONS: NOVEMBER 30: INTEGRATION

MATH 152, SECTION 55 (VIPUL NAIK)

1. Performance review

11 people took this 7-question quiz. The score distribution was as follows:

- Score of 2: 1 person
- Score of 3: 2 people
- Score of 4: 2 people
- Score of 5: 2 people
- Score of 6: 4 people

The mean score was 4.17 out of 7. Here are the problem answers:

(1) (A): 7 people
(2) (B): 8 people
(3) (C): 6 people
(4) (D): 8 people
(5) (A): 10 people
(6) (B): 5 people
(7) (B): 6 people

2. Solutions

(1) What is the limit $\lim_{x\to\infty}\left[\left(\int_0^x \sin^2\theta d\theta\right)/x\right]$?
(A) $1/2$
(B) 1
(C) $1/\pi$
(D) $2/\pi$
(E) $1/(2\pi)$

Answer: Option (A)

Explanation: The average over a period is 1/2. Thus, this is also the limit of the average over intervals of arbitrarily large length. See the notes on the mean value of a periodic function over a period.

Performance review: 7 out of 11 got this correct. 3 chose (B) and 1 chose (D).

Historical note (last year): 13 out of 16 people got this correct. 1 person each chose (C) and (E) and 1 person left the question blank.

(2) Consider the substitution $u = -1/x$ for the integral $\int \frac{dx}{x^2+1}$. What is the **new integral**?
(A) $\int \frac{du}{u(u^2+1)}$
(B) $\int \frac{du}{u^2+1}$
(C) $\int \frac{udu}{u^2+1}$
(D) $\int \frac{u^2du}{u^2+1}$
(E) $\int \frac{u^2du}{(u^2+1)^2}$

Answer: Option (B)

Explanation: Setting $u = -1/x$, we get $x = -1/u$, so $dx/du = 1/u^2$. Plugging in, we get:

$$\int \frac{(1/u^2)\,du}{(-1/u)^2+1} = \int \frac{du}{1+u^2}$$

Performance review: 8 out of 11 got this correct. 2 chose (D), 1 chose (A).

Historical note (last year): 8 out of 16 people got this correct. 6 people chose (D), 1 person chose (C), and 1 person left the question blank.

Action point: Those who chose (D) either made a calculation error or forgot the relative derivative $1/u^2$ in the numerator. Please make sure you review u-substitutions and get this kind of question correct in the future.

(3) *Hard*: What is the **value** of $c \in (0, \infty)$ such that $\int_0^c \frac{dx}{x^2+1} = \lim_{a\to\infty} \int_c^a \frac{dx}{x^2+1}$?

(A) $\frac{1}{\sqrt{3}}$

(B) $\frac{1}{\sqrt{2}}$

(C) 1

(D) $\sqrt{2}$

(E) $\sqrt{3}$

Answer: Option (C)

Explanation: By the previous question, we get:

$$\lim_{a\to\infty} \int_c^a \frac{dx}{x^2+1} = \int_{-1/c}^0 \frac{du}{u^2+1}$$

Because the integrand is even, we get that:

$$\lim_{a\to\infty} \int_c^a \frac{dx}{x^2+1} = \int_0^{1/c} \frac{dx}{x^2+1}$$

Thus, from the given data, we get:

$$\int_0^c \frac{dx}{x^2+1} = \int_0^{1/c} \frac{dx}{x^2+1}$$

This gives:

$$\int_c^{1/c} \frac{dx}{x^2+1} = 0$$

But the function in question is a positive function, hence the only way the above can hold is if $c = 1/c$, giving $c = 1$ (since c is positive).

Note that the problem can also be solved using the "fact" that arctan is an indefinite integral, so we note that the integral from 0 to 1, as well as the integral from 1 to ∞, are both $\pi/4$. However, that solution requires a knowledge of the antiderivative and of the properties of inverse trigonometric functions, whereas this proof does not require any development of that theory.

Performance review: 6 out of 11 got this correct. 3 chose (E), 1 chose (B), 1 chose (D).

Historical note (last year): 8 out of 16 people got this correct. 6 people chose (B), for reasons unclear. 1 person chose (A) and 1 person left the question blank.

(4) Suppose f is a continuous nonconstant even function on $\mathbb{R}$. Which of the following is **true**?

(A) Every antiderivative of f is an even function.

(B) f has exactly one antiderivative that is an even function.

(C) Every antiderivative of f is an odd function.

(D) f has exactly one antiderivative that is an odd function.

(E) None of the antiderivatives of f is either an even or an odd function.

Answer: Option (D)

Explanation: The odd function will be the unique antiderivative that takes the value 0 at 0. Specifically, if F is an antiderivative of f, we can easily check that:

$$F(x) - F(0) = \int_0^x f(t)\,dt = \int_{-x}^0 f(t)\,dt = F(0) - F(-x)$$

Thus, we get that:

$$F(0) = \frac{F(x) + F(-x)}{2}$$

Thus, F has half-turn symmetry about $(0, F(0))$. It is odd iff $F(0) = 0$. We see that, among the family of antiderivatives, there is a unique one with the property.

Additional note: A little while ago, you proved that a cubic function enjoys half-turn symmetry about its point of inflection, and were supposed to give a computational proof thereof. The fact can actually be deduced without any computation using the fact that the derivative function, the quadratic, has *mirror symmetry* about the same x-value. The proof of that fact follows the same lines as the proof given above.

Performance review: 8 out of 11 got this correct. 3 chose (C).

Historical note (last year): 4 out of 16 people got this correct. 9 people chose (C), apparently forgetting the fact that an odd function must be 0 at 0. 1 person chose (A) and 2 people chose (B).

(5) Suppose f is a continuous nonconstant odd function on $\mathbb{R}$. Which of the following is **true**?
(A) Every antiderivative of f is an even function.
(B) f has exactly one antiderivative that is an even function.
(C) Every antiderivative of f is an odd function.
(D) f has exactly one antiderivative that is an odd function.
(E) None of the antiderivatives of f is either an even or an odd function.

Answer: Option (A)

Explanation: Fill this in yourself; it is similar to the previous exercise. Note that the key difference here is that an even function does not have to be 0 at 0, and adding a constant preserves the property of being even.

Performance review: 10 out of 11 got this correct. 1 chose (E).

Historical note (last year): 13 out of 16 people got this correct. 1 person each chose (B), (C), and (D).

(6) Suppose f is a continuous nonconstant periodic function on $\mathbb{R}$ with period h. Which of the following is **true**?
(A) Every antiderivative of f is a periodic function with period h, regardless of the choice of f.
(B) For some choices of f, every antiderivative of f is a periodic function; for all others, f has no periodic antiderivative.
(C) f has exactly one periodic antiderivative for every choice of f.
(D) For some choices of f, f has exactly one periodic antiderivative; for all others, f has no periodic antiderivative.
(E) Regardless of the choice of f, no antiderivative of f can be periodic.

Answer: Option (B)

Explanation: What this crucially depends on is the mean value of f over a period. If this mean value is 0 (e.g., for sin and cos), then every antiderivative is periodic. If the mean value is nonzero (e.g., $x \mapsto 1 + \sin x$ or $\sin^2$) then the antiderivative is (linear + periodic), and that mean value is the slope of the linear component of any antiderivative. For instance, $1 + \cos x$ has mean value 1, and its antiderivative, $x + \sin x$, has linear part x of slope 1 and periodic part $\sin x$.

Performance review: 5 out of 11 got this correct. 4 chose (A), 2 chose (D).

Historical note (last year): 5 out of 16 people got this correct. 9 people chose (A), suggesting that they didn't remember the ideas about non-periodic functions with periodic derivatives. 2 people chose (A).

Action point: Review the material on functions that are "periodic with shift" – discussed when we covered graphing of functions.

(7) Consider a continuous increasing function f defined on the nonnegative real numbers. Define $m_f(a)$, for $a > 0$, as the unique value $c \in [0, a]$ such that $f(c)$ is the mean value of f on the interval $[0, a]$.

If $f(x) := x^n$, n an integer greater than 1, what kind of function is m_f (your answer should be valid for all n)?
(A) $m_f(a)$ is a constant λ dependent on n but independent of a.
(B) It is a function of the form $m_f(a) = \lambda a$, where λ is a constant depending on n.

(C) It is a function of the form $m_f(a) = \lambda a^{n-1}$, where λ is a constant depending on n.
(D) It is a function of the form $m_f(a) = \lambda a^n$, where λ is a constant depending on n.
(E) It is a function of the form $m_f(a) = \lambda a^{n+1}$, where λ is a constant depending on n.

Answer: Option (B)

Explanation: The integral on the interval $[0, a]$ is $a^{n+1}/(n+1)$. The mean value is $a^n/(n+1)$. The value c is thus $(a^n/(n+1))^{1/n} = a/(n+1)^{1/n}$. Setting $\lambda = 1/(n+1)^{1/n}$, we see that option (B) works.

Performance review: 6 out of 11 got this correct. 4 chose (C), 1 chose (D).

Historical note (last year): 1 out of 16 people got this correct. 5 people chose (D), 4 people chose (E), 3 people chose (C), 2 people chose (A), and 1 person left the question blank. Most probably, people forgot the step of raising to the power of $1/n$, and of course, many people just guessed.

Action point: Make sure that you can solve the problem under fewer time constraints.