LOGARITHM, EXPONENTIAL, DERIVATIVE, AND INTEGRAL

MATH 152, SECTION 55 (VIPUL NAIK)

Corresponding material in the book: Section 7.2, 7.3, 7.4.

What students should definitely get: The definition of logarithm as an integral, its key properties. The differentiation and integration formulas for logarithm and exponential, the key ideas behind combining these with the chain rule and u-substitution to carry out other integrals.

Executive summary

0.1. Logarithm and exponential: basics.

(1) The *natural logarithm* is a one-to-one function with domain $(0, \infty)$ and range $\mathbb{R}$, and is defined as $\ln(x) := \int_1^x (dt/t)$.
(2) The natural logarithm is an increasing function that is concave down. It satisfies the identities $\ln(1) = 0$, $\ln(ab) = \ln(a) + \ln(b)$, $\ln(a^r) = r \ln a$, and $\ln(1/a) = -\ln a$.
(3) The limit $\lim_{x \to 0} \ln(x)$ is $-\infty$ and the limit $\lim_{x \to \infty} \ln(x)$ is $+\infty$. Note that ln goes off to $+\infty$ at ∞ even though its derivative goes to zero as $x \to +\infty$.
(4) The derivative of $\ln(x)$ is $1/x$ and the derivative of $\ln(kx)$ is also $1/x$. The derivative of $\ln(x^r)$ is r/x.
(5) The antiderivative of $1/x$ is $\ln|x| + C$. What this really means is that the antiderivative is $\ln(-x) + C$ when x is negative and $\ln(x) + C$ when x is positive. If we consider $1/x$ on both positive and negative reals, the constant on the negative side is unrelated to the constant on the positive side.
(6) e is defined as the unique number x such that $\ln(x) = 1$. e is approximately 2.718. In particular, it is between 2 and 3.
(7) The inverse of the natural logarithm function is denoted exp, and $\exp(x)$ is also written as e^x. When x is a rational number, $e^x = e^x$ (i.e., the two definitions of exponentiation coincide). In particular, $e^1 = e$, $e^0 = 1$, etc.
(8) The function exp equals its own derivative and hence also its own antiderivative. Further, the derivative of $x \mapsto e^{mx}$ is me^{mx}. Similarly, the integral of e^{mx} is $(1/m)e^{mx} + C$.
(9) We have $\exp(x + y) = \exp(x)\exp(y)$, $\exp(rx) = (\exp(x))^r$, $\exp(0) = 1$, and $\exp(-x) = 1/\exp(x)$. All of these follow from the corresponding identities for ln.

Actions...

(1) We can calculate $\ln(x)$ for given x by using the usual methods of estimating the values of integrals, applied to the function $1/x$. We can also use the known properties of logarithms, as well as approximate ln values for some specific x values, to estimate $\ln x$ to a reasonable approximation. For this, it helps to remember $\ln 2$, $\ln 3$, and $\ln 5$ or $\ln 10$.
(2) Since both ln and exp are one-to-one, we can *cancel* ln from both sides of an equation and similarly *cancel* exp. Technically, we cancel ln by applying exp to both sides, and we cancel exp by applying ln to both sides.

0.2. Integrations involving logarithms and exponents.

Words/actions ...

(1) If the numerator is the derivative of the denominator, the integral is the logarithm of the (absolute value of) the denominator. In symbols, $\int g'(x)/g(x)\,dx = \ln|g(x)| + C$.
(2) More generally, whenever we see an expression of the form $g'(x)/g(x)$ inside the integrand, we should consider the substitution $u = \ln|g(x)|$. Thus, $\int f(\ln|g(x)|)g'(x)/g(x)\,dx = \int f(u)\,du$ where $u = \ln|g(x)|$.
(3) $\int f(e^x)e^x\,dx = \int f(u)\,du$ where $u = e^x$.
(4) $\int e^x[f(x) + f'(x)]\,dx = e^x f(x) + C$.

(5) $\int e^{f(x)} f'(x)\,dx = e^{f(x)} + C$.
(6) Trigonometric integrals: $\int \tan x\,dx = -\ln|\cos x| + C$, and similar integration formulas for cot, sec and csc: $\int \cot x\,dx = \ln|\sin x| + C$, $\int \sec x = \ln|\sec x + \tan x| + C$, and $\int \csc x\,dx = \ln|\csc x - \cot x| + C$.

1. Logarithms: the adventure begins

1.1. **Finding an antiderivative of the reciprocal function.** Recall that the process of differentiation never gave us fundamentally new functions, because the derivatives of all the basic functions that we knew were expressible in terms of other basic functions, and using the operations of pointwise combination and composition did not allow us to break ground into new functions. The situation differs somewhat for integration. We have seen that we often come across functions for which we have no clue as to how to find an antiderivative. We now discuss how to handle one such function.

This function is the function $1/x$, which, for now, we will assume to be a function on $(0, \infty)$. We want to find an antiderivative for this function.

The basic results of integration tell us that one way of defining an antiderivative is by using a definite integral from a fixed value to x, as long as that fixed value is in the domain. For reasons that are not immediately obvious, we choose the fixed value (the reference point) as 1. We thus define the following function:

$$L(x) := \int_1^x \frac{dt}{t}$$

Note that this is the *unique* antiderivative which has the property that its value at 1 is 0. By definition, $L'(x) = 1/x$ for all x. What further information can we derive about L?

1.2. **Using the multiplicative transform.** By the u-substitution method, we can readily verify that, for $a, b > 0$:

$$\int_1^a \frac{dt}{t} = \int_b^{ab} \frac{dt}{t}$$

The key thing that is special about $1/x$ is that the multiplicative factor on the dt part cancels the multiplicative factor on the t part.

This gives us that:

$$L(a) - L(1) = L(ab) - L(b)$$

Since $L(1) = 0$, we obtain that L is a function satisfying the property:

$$L(ab) = L(a) + L(b) \ \forall\ a, b > 0$$

Thus, even though we do not have an explicit description of L, we know that L converts products to sums. In particular, we also see, for instance, that:

$$L(a^n) = nL(a) \ \forall a > 0, n \in \mathbb{Z}$$

In particular, $L(1/a) = -L(a)$.

We can further see that for any rational number r, we have:

$$L(a^r) = rL(a) \ \forall a > 0, r \in \mathbb{Q}$$

In other words, the function L converts products to sums and pulls the exponent into a multiple. We also know that since $L'(x) > 0$ for all $x > 0$, L is continuous and increasing. In particular, we see that L is a one-to-one map on $(0, \infty)$.

What is the range of L? Consider $a = 2$. Then, $L(a) = L(2) > 0$. As $n \to \infty$, $L(a^n) = nL(a) \to \infty$, and as $n \to -\infty$, $L(a^n) = nL(a) \to -\infty$. Since L is increasing, we can use this to see that $\lim_{x\to\infty} L(x) = \infty$ and $\lim_{x\to 0} L(x) = -\infty$. Further, by the intermediate value theorem, we see that the range of L is $\mathbb{R}$.

The upshot: L is a continuous increasing one-to-one function from $(0, \infty)$ to $\mathbb{R}$ that sends 1 to 0 and converts products to sums.

1.3. **L for (natural) logarithm.** The function L that is described above is termed the *natural logarithm* function. It is ubiquitous in mathematics, and is denoted ln. Thus, we have the definition:

$$\ln x := \int_1^x \frac{dt}{t} \ \forall\ x > 0$$

It turns out that this natural logarithm behaves in ways very similar to logarithms to base 10. A quick primer for those who didn't live in prehistoric times: in the olden days, when people had to do multiplications by hand, they used a tool called *logarithm tables* to do these multiplications. The logarithm tables basically converted the multiplication problem to an addition problem.

Here is the principle on which the logarithm tables worked. These tables allowed you to, for a given number x, find the approximate value of r such that $10^r = x$. This value of r is called $\log_{10} x$. Then, if you had to multiply x and y, you first found $\log_{10} x$ and $\log_{10} y$. It turns out that $\log_{10}(xy) = \log_{10}(x) + \log_{10} y$, because if $10^r = x$ and $10^s = y$, then $10^{r+s} = xy$ by properties of exponents. Thus, to find xy, we find $\log_{10} x$ and $\log_{10}(y)$ and add them. Then, there are antilogarithm tables, that allow us to find the antilogarithm of this sum that we have computed (or basically, raise 10 to the power of that number).

The principle of logarithm tables was later converted to a *mechanical device* called the *slide rule*. How many people have used slide rules? What a slide rule does is use a *logarithmic scale*, i.e., it places numbers on a scale in such a way that the distance between the positions of two numbers is determined by their quotient. So, on a logarithmic scale, the distance between 1 and 10 is the same as the distance between 10 and 100, and also the same as the distance between 0.01 and 0.1. The distance between 3 and 7 is the same as the distance between 30 and 70. (If you're interested in pictures of slide rules, do a Google image search. I haven't included any picture here because of potential copyright considerations).

A slide rule comprises two logarithmic scales (using the same calibration) but one of them can slide against each other. We can use the sliding scale to add lengths along the scale, but since the scale is logarithmic, this ends up multiplying the numbers. You may have heard about how people with an abacus can often do simple calculations faster than people with a calculator. It turns out that people with a slide rule can usually do multiplications faster than people with a calculator.[1]

Logarithmic scales are used in many measurements. Here are some examples:

(1) The **Richter magnitude scale** measures the intensity of earthquakes. It is calibrated logarithmically to base 10. An earthquake one point higher on the Richter scale is ten times as intense.
(2) The **pH scale** in chemistry is a logarithmic scale to measure the concentration of the H^+ (more precisely, H_3O^+) ions. It is a negative logarithmic scale to base 10. An increase in the pH value by 1 corresponds to a decrease in the hydronium ion concentration to 1/10 of its original value.
(3) The **decibel scale**, used for sound levels and other level measurements, is a logarithmic scale where an increase in 10 points along the scale corresponds to a ten-fold increase in amplitude. Thus, $20dB$ is ten times as loud as $10dB$.

The natural logarithm function can be thought of as creating a logarithmic scale on the positive reals. But the question we are concerned with is: what precisely is this scale? How does this compare with the usual logarithm to base 10? Our hunch is that there should be a number e such that $\ln(x)$ is the value r such that $x = e^r$. What must this number e be?

2. The back and forth of things: logarithm and exponential

2.1. **In search of e.** If such a number e exists, then it must be the unique number x satisfying $\ln(x) = 1$. Further, since ln is an increasing function, we can try locating e between two consecutive integers by determining $\ln 2$, $\ln 3$, and so on. Actually, we can be more clever.

We can begin by trying to compute $\ln 2$. We could do this using upper and lower sums. We could also do it by noting that $\ln x = \int_1^x dt/t$, and must be located between the antiderivatives of $x^{-1/2}$ and $x^{-3/2}$. We did this approximation a few weeks ago and found that $\ln 2$ is located between 0.58 and 0.83. Further approximations using either this method or upper and lower sums for partitions yields than $\ln 2$ is between 0.69 and 0.70. We will assume $\ln 2 \approx 0.7$ for calculations.

[1]On the other hand, the calculator is a lot more versatile than the slide rule, and is probably faster for computing roots, multiplying long sequences of numbers, or combinations of multiplication and addition.

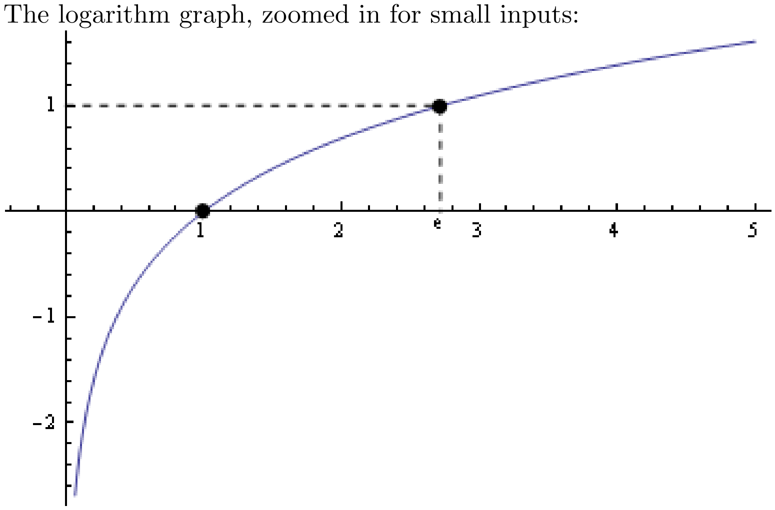
Now that we know $\ln 2$, we do not need to do any more messy work with upper and lower sums. We know that $2 < 2\sqrt{2} < 3$. We also have that $\ln(2\sqrt{2}) = (3/2)\ln 2 \approx 1.05$, which is bigger than 1. Thus, $\ln 3 > 1$, so $2 < e < 3$. In fact, $2 < e < 2\sqrt{2} \approx 2.83$. We can also calculate, for instance, that the cuberoot of 2 is about 1.26. Thus, $\ln(2.52)$ is approximately $(4/3)(\ln 2)$ which is approximately 0.93. Thus, we get that e should be bigger than 2.52. A similar process of successive approximations yields that the value of e is approximately 2.718281828. You should know e to at least three decimal places: 2.718.

Clearly, since $\ln(e) = 1$, we have that $\ln(e^{p/q}) = p/q$ for integers p and q, with $q \neq 0$. It is not clear a priori what we would mean by the notation e^r for irrational numbers r, but whatever we may mean, it should be the case that $\ln(e^r) = r$. In other words, the function $x \mapsto e^x$ must be the inverse function of the one-to-one function ln. The function $x \mapsto e^x$ is also called the exponentiation function, and sometimes denoted exp. By the way, the letter e could be thought of as standing for *exponentiation*, but historically it is believed to be named after Leonhard Euler, a prolific mathematician who studied the properties both of the number e and the exponentiation function.

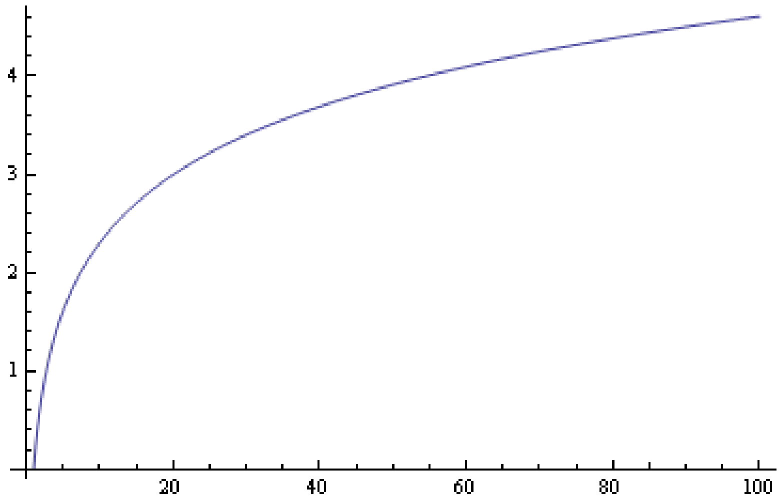
2.2. **Logarithms and exponents: some rules.** Here are some of the rules:

(1) ln is a one-to-one function from $(0, \infty)$ to $\mathbb{R}$ and exp is a one-to-one function from $\mathbb{R}$ to $(0, \infty)$. The two functions are inverse functions of each other.
(2) ln converts products to sums and exp converts sums to products. In other words, $\ln(xy) = \ln(x) + \ln(y)$ and $\exp(x + y) = \exp(x)\exp(y)$.
(3) $\ln(1) = 0$ and $\exp(0) = 1$.
(4) $\ln(1/x) = -\ln x$ and $\exp(-x) = 1/\exp(x)$.
(5) $\ln(x^r) = r \ln x$ and $\exp(rx) = (\exp(x))^r$.
(6) Both exp and ln are continuous and increasing functions.
(7) exp has the x-axis as a horizontal asymptote as $x \to -\infty$, while ln has the y-axis as a vertical asymptote as $x \to 0$.
(8) exp is concave up and ln is concave down.

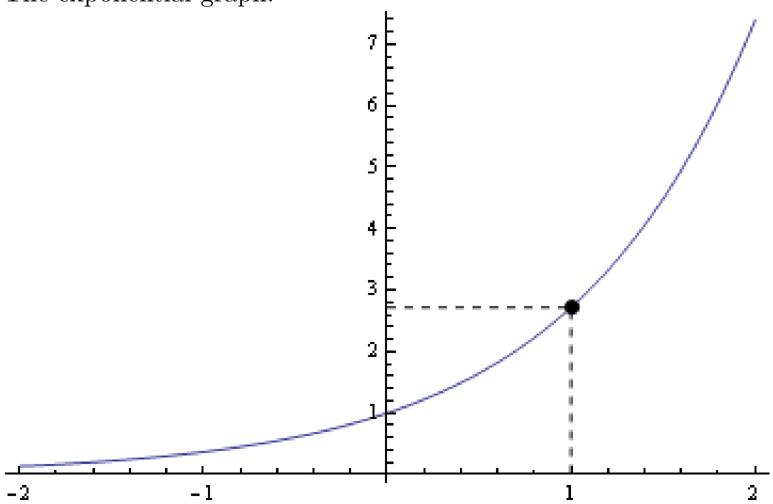
Here are the graphs:
The logarithm graph, zoomed in for small inputs:



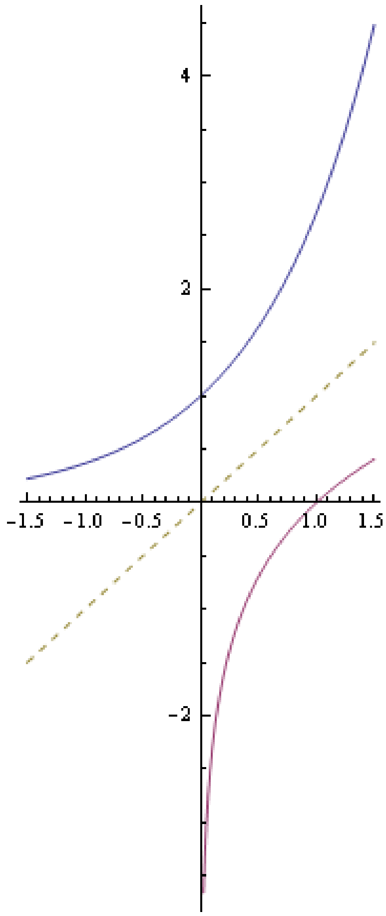
The logarithm graph, zoomed out:



The exponential graph:



Here are the logarithm and exponential graphs together, so that we can see that the graphs are reflections of each other about the $y = x$ line:



2.3. **Numerical shennanigans.** In his autobiographical book, Richard Feynmann discusses how he impressed a bunch of mathematicians by being able to calculate natural logarithms of many numbers to one or two decimal places. However, what he did was hardly impressive. It turns out that remembering the natural logarithms of a few numbers allows us to compute them approximately for many numbers.

For instance, it is useful to remember that $\ln(2) \approx 0.6931$, $\ln(3) \approx 1.0986$, $\ln(10) \approx 2.3026$, and $\ln(7) \approx 1.9459$. We can now calculate the logarithm values for most integers. How? Using the fact that logarithms translate multiplication to addition. Thus, $\ln(4) = 2\ln(2) \approx 1.3862$, while $\ln(5) = \ln(10) - \ln(2) \approx 1.6095$. In fact, we can readily calculate the natural logarithm of any positive integer all of whose prime factors are among 2, 3, 5, and 7. What about $\ln(11)$? While we cannot calculate this precisely, we can calculate $\ln(10)$ and $\ln(12)$ and thus obtain reasonable upper and lower bounds for $\ln(11)$. Even better, we know that $\ln(120) < 2\ln(11) < \ln(125)$, and since we can calculate both $\ln(120)$ and $\ln(125)$, we get a pretty small range for $\ln(11)$.

In fact, Feynman was able to impress physicists by doing calculations that essentially relied on only two facts: $\ln(2) \approx 0.7$ and $\ln(10) \approx 2.3$.

If we are able to quickly calculate natural logarithms, a happy corollary of that is that we can quickly integrate dx/x on intervals.

2.4. **Domain and range issues.** For domain computations in the past, we used the following basic guidelines:

(1) Things in the denominator must be nonzero.
(2) Things with squareroots or even roots must be nonnegative.
(3) Things with squareroots or even roots in the denominator must be positive.

We now add two more criteria:

(4) Things under logarithm must be positive.
(5) Things under logarithm of the absolute value must be nonzero.

2.5. **Logarithm of the absolute value.** The natural logarithm function is defined only for positive reals. However, we can extend it to a function on all reals by taking the absolute value first, i.e., we look at the function $x \mapsto \ln(|x|)$. This is an even function and its graph is obtained by takingthe graph of the logarithm function and adding its mirror image about the y-axis

It turns out that the derivative of $\ln(|x|)$ is $1/x$. In particular, we see that $\ln(|x|)$ serves as an antiderivative of $1/x$ for *all nonzero* x. This is an improvement on $\ln(x)$, which worked only for positive x. However, we should be careful because the domain of $1/x$ as well as of $\ln |x|$ excludes zero. Hence, the behavior on the positive and negative side are totally independent of each other. We shall return to this point in a later lecure.

3. Formulas for derivatives and integrals

3.1. **Derivative and integral formulas for logarithms.** The main formula that we have, which follows from our definition of natural logarithm, is the following:

$$\frac{d}{dx}(\ln x) = \frac{1}{x}$$

This is the formula for $x > 0$. A more general version, for $x \neq 0$, is:

$$\frac{d}{dx}(\ln |x|) = \frac{1}{x}$$

The corresponding antiderivative formula for $x > 0$ is:

$$\int \frac{dx}{x} = \ln(x) + C$$

In general, the antiderivative formula is:

$$\int \frac{dx}{x} = \ln(|x|) + C$$

However, it should be remembered that this formula is valid only when we are working with x either in $(0, \infty)$ or in $(-\infty, 0)$, i.e., we cannot use the formula to cross between the interval $(0, \infty)$ and $(-\infty, 0)$. In fact, if we have a function $f : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ such that $f'(x) = 1/x$, then we can guarantee that $f(x) - \ln(|x|)$ is constant on $x > 0$ and is constant on $x < 0$. However, these constants may differ. The behavior on the $(0, \infty)$ connected component does not in any way constrain the behavior on the $(-\infty, 0)$ connected component.

3.2. **The exponential and its derivative and integral formulas.** Recall that exp is the inverse of the ln function. Thus, we can use the rule for differentiating the inverse function to find $\exp'$. We have:

$$\exp'(x) = \frac{1}{\ln'(\exp x)} = \frac{1}{\frac{1}{\exp(x)}} = \exp(x)$$

Thus, we have the remarkable property that the exponential function, i.e., the function $x \mapsto e^x$, is its own derivative. Another way of thinking about this is that the rate of growth of the exponential function is *equal* to its value. Note that this also implies that the exponential function is infinitely differentiable and all higher derivatives equal the same function.

We rewrite the above in Liebniz notation:

$$\frac{d}{dx}(e^x) = e^x$$

We also note the corresponding statement for indefinite integration:

$$\int e^x \, dx = e^x + C$$

3.3. **Some corollaries.** Using the above, we obtain the following identities for the logarithm and exponent:

$$\begin{aligned}
\frac{d}{dx}(\ln(kx)) &= \frac{1}{x} \\
\frac{d}{dx}(\ln(x^r)) &= \frac{r}{x} \\
\frac{d}{dx}(e^{mx}) &= me^{mx} \\
\int e^{mx}\,dx &= \frac{1}{m}e^{mx} + C
\end{aligned}$$

Each of these identities can be derived in two ways: either by using the properties of logarithms and exponents on the inside and then differentiating, or by first differentiating and then simplifying. For instance, for the second identity, we can either simplify $\ln(x^r)$ as $r\ln(x)$ first and then pull the constant r out before differentiating, or we can use the chain rule to obtain $(1/x^r)\cdot rx^{r-1}$. It is gratifying to know that the answers we obtain both ways are the same.

4. Application to indefinite and definite integration

4.1. **The u-substitution: a textbook example.** We begin with an easy example:

$$\int_{\sqrt{n}}^{n} \frac{1}{x\ln x}\,dx$$

Here, n is an integer greater than 1.

Believe it or not, this integral actually came up in some asymptotic approximations I was doing some time ago to figure out whether some numbers have large prime divisors! Let us first look at the indefinite integral. The substitution $u = \ln x$ gives us:

$$\int \frac{1}{x\ln x} = \int \frac{du}{u} = \ln(u) = \ln(\ln x) + C$$

Note that we do not need to put absolute values here because on the interval of integration, ln is positive. Now, we can evaluate between limits:

$$[\ln(\ln x)]_{\sqrt{n}}^{n} = \ln(\ln n) - \ln(\ln\sqrt{n}) = \ln\left[\frac{\ln n}{\ln\sqrt{n}}\right] = \ln\left[\frac{\ln n}{(1/2)\ln n}\right] = \ln 2$$

So, the answer is $\ln 2$, which, as we computed earlier, is approximate 0.693. Apparently, this is the rough heuristic argument for why about 69.3% of the numbers have a prime divisor greater than their squareroot.

4.2. **Numerator as derivative of denominator.** The gist of this logarithmic substitution can be captured by the formula:

$$\int \frac{g'(x)}{g(x)}\,dx = \ln|g(x)| + C$$

The proof of this proceeds via setting $u = g(x)$. Thus, the general idea when using logarithmic substitutions is to try to obtain the numerator as the derivative of the denominator. For instance, consider the integral:

$$\int \frac{x}{x^2+1}\,dx$$

Here, the derivative of the denominator is $2x$, so we adjust by a factor of 2 to obtain:

$$\frac{1}{2}\int \frac{2x}{x^2+1}\,dx = \frac{1}{2}\ln(|x^2+1|) + C$$

Note that in this case, since x^2+1 is always positive, the absolute value can be dropped and we get $\frac{1}{2})(x^2+1) + C$. Further, this antiderivative is valid over all reals.

4.3. **Trigonometric integrals involving logarithms.** Recall so far that we have seen the antiderivatives of sin, cos, $\sec^2$, $\sec\cdot\tan$, $\csc^2$, and $\csc\cdot\cot$. We also used these, along with trigonometric identities, to compute antiderivatives for $\sin^2$, $\cos^2$, $\tan^2$, and $\cot^2$. All these results were obtained as corollaries of the differentiation formulas.

We now try to obtain a formula to integrate cot. The key idea is to note that:

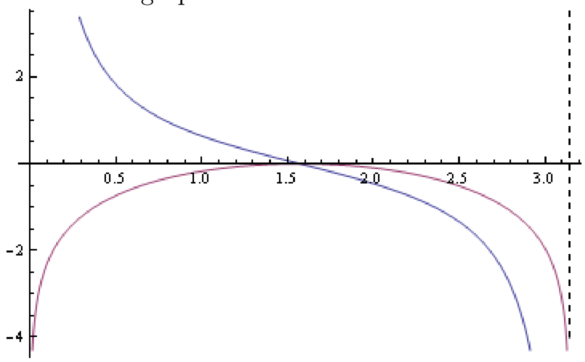
$$\cot x = \frac{\cos x}{\sin x}$$

Since cos is the derivative of sin, this matches up with the general pattern that we just discussed, and we obtain that the antiderivative of cot is $\ln \circ |\sin|$. In other words:

$$\int \cot x \, dx = \ln|\sin x| + C$$

Note that cot is undefined at multiples of π, and so any integration of this sort is valid only if the entire interval of integration lies strictly between two consecutive multiples of π. It is also instructive to graph the antiderivative of cot.

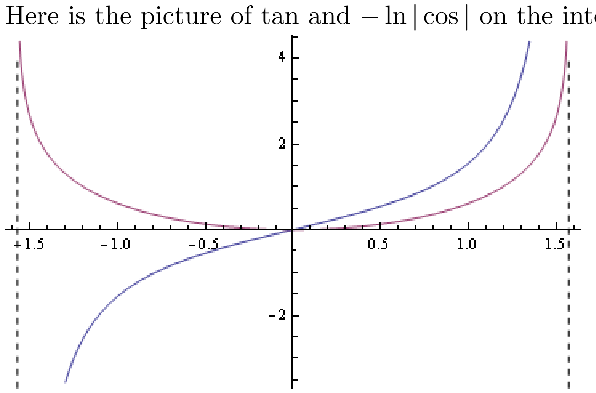
Here is the graph of cot and its antiderivative on the interval $(0, \pi)$, where both are defined:



Similarly, we obtain:

$$\int \tan x \, dx = -\ln|\cos x| + C = \ln|\sec x| + C$$

Here is the picture of tan and $-\ln|\cos|$ on the interval $(-\pi/2, \pi/2)$, where both are defined:



Note that those two expressions are the same because cos and sec are reciprocals of each other.

Let us look at a somewhat harder integral: the integral of the secant function:

$$\int \sec x \, dx = \int \frac{\sec x(\sec x + \tan x)}{\sec x + \tan x} \, dx = \int \frac{\sec^2 x + \sec x \tan x}{\tan x + \sec x} \, dx$$

The numerator is the derivative of the denominator, and we obtain:

$$\int \sec x \, dx = \ln|\sec x + \tan x| + C$$

In a similar vein, we obtain that:

$$\int \csc x \, dx = \ln |\csc x - \cot x| + C$$

4.4. **Domain and range issues.** When doing indefinite integration, it is often best to forget about issues of domain and range and just let the algebraic manipulations flow. However, to interpret the results at the end, it is important to look at the domain and range issues. Ideally, the antiderivative should be defined and should make sense on all intervals where the function itself is continuous. Further, if there are points where the function is continuous but the *expression obtained for the antiderivative* is not defined, we should try to obtain the limit at that point.

4.5. **An application: integrating the cube of the tangent function.** Let us look at an application of the above:

$$\int \tan^3 x \, dx$$

Before we proceed, it is worth remarking how different integration is from differentiation. For differentiation, there was just the formula for differentiating sin and cos, and everything else followed using the product rule and quotient rule. We still memorized more, but that was mainly to speed things up, not out of necessity. With integration, on the other hand, we need to have a whole bag of *ad hoc* tricks that we try one after the other.

Let us look at this integral. The key thing to do here is to break down $\tan^3 x = \tan x \cdot \tan^2 x$. Next, we use $\tan^2 x = \sec^2 x - 1$, and we have:

$$\int \tan x \sec^2 x \, dx - \int \tan x \, dx$$

The first integral can be quickly calculated using the chain rule or u-substitution, since $\sec^2 x$ is the derivative of $\tan x$. The second integral comes from our formula, and we get:

$$\frac{\tan^2 x}{2} + \ln |\cos x| + C$$

4.6. **A fancier formula.** Here is a formula that uses the chain rule twice:

$$\int \frac{g'(x) f(\ln |g(x)|)}{g(x)} \, dx = \int f(u) \, du$$

where $u = \ln(|g(x)|)$. For instance:

$$\int \frac{2x(\ln(x^2+1))^3}{x^2+1} \, dx = \int u^3 \, du$$

where $u = \ln(x^2 + 1)$. This further simplifies to:

$$\frac{1}{4}[\ln(x^2+1)]^4 + C$$

5. More tricks and techniques

5.1. **Logarithmic differentiation.** Logarithmic derivatives are both a conceptual and a computational tool. Currently, we focus on the computational aspects. The idea is to use the same formula that we obtained earlier, but in reverse:

$$\frac{d}{dx} \ln(|g(x)|) = \frac{g'(x)}{g(x)}$$

Rearranging the terms yields:

$$g'(x) = g(x) \frac{d}{dx} \ln(|g(x)|)$$

If g is a product of functions $g_1, g_2, \ldots, g_n$, then $\ln|g(x)| = \ln|g_1(x)| + \cdots + \ln|g_n(x)|$, and we get:

$$g'(x) = g(x)\left[\frac{g_1'(x)}{g_1(x)} + \frac{g_2'(x)}{g_2(x)} + \cdots + \frac{g_n'(x)}{g_n(x)}\right]$$

Note that this expression is not *really* new and did not *really* require logarithms. You can in fact convince yourself that it is just a reformulation of the product rule. When $g = g_1 g_2$, for instance, this says that:

$$g' = g\left[\frac{g_1'}{g_1} + \frac{g_2'}{g_2}\right]$$

Substituting $g = g_1 g_2$, this simplifies to the usual product rule. However, the logarithmic formulation has some conceptual advantages.

For instance, suppose $g(x) := x(x-1)(x-2)$. Then, we immediately obtain that:

$$\frac{g'(x)}{g(x)} = \frac{1}{x} + \frac{1}{x-1} + \frac{1}{x-2}$$

Further, of we have $g(x) = g_1(x)^{a_1} g_2(x)^{a_2} \ldots g_n(x)^{a_n}$, we obtain that:

$$\frac{g'(x)}{g(x)} = \frac{a_1 g_1'(x)}{g_1(x)} + \frac{a_2 g_2'(x)}{g_2(x)} + \cdots + \frac{a_n g_n'(x)}{g_n(x)}$$

So, if $g(x) = x^3(x-1)^4(x-2)^5$, we obtain that:

$$\frac{g'(x)}{g(x)} = \frac{3}{x} + \frac{4}{x-1} + \frac{5}{x-2}$$

5.2. **Exponentiation tricks.** We have already seen basic integration and differentiation identities for the exponentiation function. There are some ways of combining these identities with the chain rule. I note some special cases here.

(1) For any function f, the derivative of $f(x)e^x$ is $(f(x) + f'(x))e^x$. Thus, the integral of $g(x)e^x$ is $f(x)e^x + C$ where $f + f' = g$.
(2) (1) is particularly useful when integrating polynomial function times e^x. This is because we can use linear algebra to find, for a given polynomial g, the unique polynomial f such that $f + f' = g$.
(3) The integral $\int f(e^x)e^x\,dx$ is $\int f(u)\,du$ where $u = e^x$.
(4) The integral $e^{f(x)} f'(x)\,dx$ is $e^{f(x)} + C$.

Let us consider an example to illustrate this. Consider the function:

$$F(x) := (x^2 + 5x + 1)e^x$$

The derivative of this, by the product rule, turn out to be e^x times the sum of x^2+5x+1 and its derivative, giving:

$$F'(x) = (x^2 + 7x + 6)e^x$$

Note that this is a new polynomial times e^x. An intersting question would be how we could reverse this procedure, i.e., given $g(x)e^x$ where g is a polynomial, how do we find a polynomial f such that the derivaitve of $f(x)e^x$ is $g(x)e^x$? By the product rule, we obtain that:

$$g(x) = f(x) + f'(x)$$

Thus, we need to find the coefficients of f. Let us do this in our concrete case where $g(x) = x^2 + 7x + 6$.

We know that the degree of f' is strictly smaller than the degree of f, so $f + f'$ has the same degree and same leading coefficient as f. In this case, this forces f to be a quadratic polynomial of the form $x^2 + mx + n$. We then get $f'(x) = 2x + m$, and we obtain that:

$$f(x) + f'(x) = x^2 + (m+2)x + (m+n)$$

Since we are given that $g(x) = x^2 + 7x + 6$, we can match coefficients and obtain:

$$m + 2 = 7, \qquad m + n = 6$$

Solving, we get $m = 5$, and $n = 1$, and we get $f(x) = x^2 + 5x + 1$, recovering our original polynomial.

Although the *specific procedure involving comparing coefficients* does require that we are *dealing with polynomials*, the general idea remains true in a broader sense: integrating $\int g(x)e^x$ is equivalent to finding a function f such that $f + f' = g$. In some cases, it is easier to think of it as an integration problem, and in others, it is easier to think of it in terms of the differental equation $f + f' = g$. The idea that these two apparently different computations measure the same thing is extremely important.